

# Higher Lie algebra actions on Lie algebroids

Marco Zambon<sup>\*</sup>, Chenchang Zhu<sup>†§</sup>

## Abstract

We consider a simple instance of action up to homotopy. More precisely, we consider strict actions of DGLAs in degrees  $-1$  and  $0$  on degree  $1$   $NQ$ -manifolds. In a more conventional language this means: strict actions of Lie algebra crossed modules on Lie algebroids.

When the action is strict, we show that it integrates to group actions in the categories of Lie algebroids and Lie groupoids (i.e. actions of  $\mathcal{LA}$ -groups and 2-groups). We perform the integration in the framework of Mackenzie's doubles.

## Contents

<b>Introduction</b>	<b>2</b>
<b>1 Infinitesimal actions</b>	<b>3</b>
1.1 Background on graded geometry . . . . .	3
1.2 Strict Lie 2-algebra actions on $NQ$ -1 manifolds . . . . .	7
<b>2 Integration to global actions</b>	<b>9</b>
2.1 Background on Mackenzie's doubles . . . . .	10
2.2 Integrating strict Lie 2-algebras: $\mathcal{LA}$ -groups and 2-groups . . . . .	12
2.3 Integrating strict actions: $\mathcal{LA}$ -group actions on Lie algebroids and 2-group actions on Lie groupoids . . . . .	15
<b>3 Examples</b>	<b>22</b>
<b>A Appendix</b>	<b>24</b>
A.1 Proof of Lemma 1.6 . . . . .	24
A.2 Proof of Prop. 2.14 . . . . .	25

---

<sup>\*</sup>Universidad Autónoma de Madrid (Dept. de Matemáticas), and ICMAT(CSIC-UAM-UC3M-UCM), Campus de Cantoblanco, 28049 - Madrid, Spain. [marco.zambon@uam.es](mailto:marco.zambon@uam.es), [marco.zambon@icmat.es](mailto:marco.zambon@icmat.es)

<sup>†</sup>Courant Research Centre "Higher Order Structures", University of Göttingen, Germany. [zhu@uni-math.gwdg.de](mailto:zhu@uni-math.gwdg.de)

<sup>‡</sup>2010 Mathematics Subject Classification: primary 53D17, 58A50, 18D35.

<sup>§</sup>Keywords: Lie algebroid,  $NQ$ -manifold, double Lie algebroid, higher Lie algebra action, higher Lie group action.

# Introduction

In recent years there has been an intense activity integrating certain infinitesimal structures, such as Lie algebroids [6][9] and  $L_\infty$ -algebras [10][11]. Here “integration” is meant in the same sense in which a Lie algebra is integrated to a corresponding Lie group. Both Lie algebroids and (non-positively graded)  $L_\infty$ -algebras are instances of *NQ-manifolds*—non-negatively graded manifolds equipped with a homological vector field (Def. 1.8). In this paper we focus on *actions*: we study infinitesimal actions on NQ-manifolds and their integrations to global actions.

We make the following elementary but important observation: While the infinitesimal symmetries of an ordinary manifold are controlled by a Lie algebra (the vector fields), the infinitesimal symmetries of a NQ-manifold  $\mathcal{M}$  are given by the *differential graded Lie algebra (DGLA)* of vector fields  $\chi(\mathcal{M})$ . Further, if  $\mathcal{M}$  is a NQ- $n$  manifold<sup>1</sup>, its infinitesimal symmetries are controlled by sub-DGLA which is an  $(n+1)$ -term DGLA—a special kind of Lie  $(n+1)$ -algebra. This implies that some kind of Lie  $(n+1)$ -group naturally plays the role of the global symmetries of  $\mathcal{M}$ .

An instance of this is given by the theory of Courant algebroids, which were a source of motivation for this work. Indeed Courant algebroids are equivalent to a special class of NQ-manifolds (symplectic NQ-2 manifolds [22]), and actions on Courant algebroids are realized by more data than just a Lie group action, as showed in [4].

In this paper we consider the simplest case  $n = 1$ , that is, NQ-1 manifolds, which in ordinary differential geometry language are just *Lie algebroids*. We saw above that the infinitesimal symmetries of NQ-1 manifolds are given by 2-term DGLAs. The latter are also known as *strict Lie 2-algebras*<sup>2</sup> (Def. 1.14), and they are equivalent to crossed modules of Lie algebras. The integration of a strict Lie 2-algebra is a strict Lie 2-group. Hence strict Lie 2-groups control the global symmetries of an NQ-1 manifold.

A *strict (infinitesimal) action* (Def. 1.15) on an NQ-1 manifold  $\mathcal{M}$  is a morphism of strict Lie 2-algebra

$$L \rightarrow \chi(\mathcal{M}).$$

In §2.3 we define the notion of *integration* of such an infinitesimal action (Def. 2.13) and we find the integrating action explicitly (Thm. 2.15, Thm. 2.16, and Prop. 2.17): it is an action

$$\mathcal{G} \times \Gamma \rightarrow \Gamma$$

which we describe explicitly, where  $\mathcal{G}$  is the strict Lie-2 group integrating  $L$  and  $\Gamma$  is the Lie groupoid integrating the Lie algebroid corresponding to  $\mathcal{M}$ .

More in detail, a strict action of  $L$  on  $\mathcal{M}$  gives rise to a “action double Lie algebroid”, which we integrate to an “action double Lie groupoid”. This is a major step, as there are no general statements in the literature which allow to integrate double Lie algebroids to double groupoids. From the “action double Lie groupoid” we extract the action  $\mathcal{G} \times \Gamma \rightarrow \Gamma$ . This gives a conceptual explanation for the results obtained in a special case by Cattaneo and the first author [8, Thm. 14.1]. Notice that the integrated action is not on  $\mathcal{M}$ , but rather

---

<sup>1</sup>This means that its coordinates are concentrated in degrees  $0, \dots, n$ .

<sup>2</sup>For a general study of Lie 2-algebras please see Baez et al. [2].

on the its integration  $\Gamma$ .

**Organization of the text:** §1 defines NQ-manifolds, strict Lie 2-algebras, and the notion of strict action. The purpose of §2 is to integrate such strict actions. We introduce the formalism of Mackenzie’s doubles, apply it to the integration of strict Lie 2-algebras (a toy example), and finally we integrate strict actions in §2.3. In §3 we give examples of Lie 2-algebra actions on tangent bundles, cotangent bundles of Poisson manifolds and Lie algebras.

**Notation and conventions:**  $M$  always denotes a smooth manifold. For any vector bundle  $E$ , we denote by  $E[1]$  the N-manifold obtained from  $E$  by declaring that the fiber-wise linear coordinates on  $E$  have degree one. If  $\mathcal{M}$  is an N-manifold, we denote by  $C(\mathcal{M})$  the graded commutative algebras of “functions on  $\mathcal{M}$ ”. By  $\chi(\mathcal{M})$  we denote graded Lie algebra of vector fields on  $\mathcal{M}$  (i.e., graded derivations of  $C(\mathcal{M})$ ).

The symbol  $A$  always denotes a Lie algebroid over  $M$ . When  $A$  is integrable, we denote the corresponding source simply connected Lie groupoid by  $\Gamma$ , and its source and target maps by  $\mathbf{s}$  and  $\mathbf{t}$ . We adopt the convention that two elements  $x, y \in \Gamma$  are composable to  $x \circ y$  iff  $\mathbf{s}(x) = \mathbf{t}(y)$ . We identify  $A \cong (\ker \mathbf{s}_*)|_M$ .

**Acknowledgements:** We thank Rajan Mehta, Tim Porter, Pavol Ševera and Jim Stasheff for their very helpful comments and for discussions. We learnt about the Artin-Mazur construction from Xiang Tang, whom we hereby thank. We are grateful to him for sharing with us this idea from [18].

Zambon thanks the Courant Research Centre “Higher Order Structures” for hospitality. Zambon was partially supported by the Centro de Matemática da Universidade do Porto, financed by FCT through the programs POCTI and POSI, by the FCT program Ciencia 2007, grants PTDC/MAT/098770/2008 and PTDC/MAT/099880/2008 (Portugal), and by MICINN RYC-2009-04065 (Spain).

Zhu thanks C.R.M. Barcelona for hospitality. Zhu is supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.

## 1 Infinitesimal actions

This section introduces NQ-manifolds (§1.1) and infinitesimal actions of Lie-2 algebras on them (§1.2). The integration of these infinitesimal actions is the main object of this paper.

### 1.1 Background on graded geometry

We start by recalling some background material on graded geometry. In §1.1.1 we consider degree 1 N-manifolds, which correspond simply to vector bundles. Then §1.1.2 we endow them with a homological vector field, obtaining Lie algebroids.

The reason why we describe classical geometric objects using graded geometry is that the latter framework allows to describe the classical structures and their symmetries in a natural way, namely in terms of vector fields. Further, since graded geometry is defined in terms of sheaves, this framework allows to make use of (graded) local coordinates.

### 1.1.1 N-manifolds

The notion of N-manifold (“N” stands for non-negative) was introduced by Ševera in [23][24]. Here we adopt the definition given by Mehta in [19, §2]. Useful references are also [5, §2][7].

If  $V = \oplus_{i < 0} V_i$  is a finite dimensional  $\mathbb{Z}_{<0}$ -graded vector space, recall that  $V^*$  is the  $\mathbb{Z}_{>0}$ -graded vector space defined by  $(V^*)_i = (V_{-i})^*$ . We use  $S^\bullet(V^*)$  to denote the *graded* symmetric algebra over  $V^*$ , so its homogeneous elements anti-commute if they both have odd degree.  $S^\bullet(V^*)$  is a graded commutative algebra concentrated in positive degrees.

Ordinary manifolds are modeled on open subsets of  $\mathbb{R}^n$ , and N-manifolds modeled on the following graded charts:

**Definition 1.1.** Let  $V = \oplus_{i < 0} V_i$  be a finite dimensional  $\mathbb{Z}_{<0}$ -graded vector space. The *local model for an N-manifold* consists of a pair as follows:

- $U \subset \mathbb{R}^n$  an open subset
- the sheaf (over  $U$ ) of graded commutative algebras given by  $U' \mapsto C^\infty(U') \otimes S^\bullet(V^*)$ .

**Definition 1.2.** An *N-manifold*  $\mathcal{M}$  consists of a pair as follows:

- a topological space  $M$  (the “body”)
- a sheaf  $\mathcal{O}_M$  over  $M$  of graded commutative algebras, locally isomorphic to the above local model (the sheaf of “functions”).

We use the notation  $C(\mathcal{M}) := \mathcal{O}_M(M)$  to denote the space of “functions on  $\mathcal{M}$ ”. By  $C_k(\mathcal{M})$  we denote the degree  $k$  component of  $C(\mathcal{M})$ , for any non-negative  $k$ . The *degree* of the graded manifold is the largest  $i$  such that  $V_{-i} \neq \{0\}$ . Degree zero graded manifolds are just ordinary manifolds:  $V = \{0\}$ , and all functions have degree zero.

**Definition 1.3.** A *vector field* on  $\mathcal{M}$  is a graded derivation of the algebra<sup>3</sup>  $C(\mathcal{M})$ .

Since  $C(\mathcal{M})$  is a graded commutative algebra (concentrated in non-negative degrees), the space of vector fields  $\chi(\mathcal{M})$ , equipped with the graded commutator  $[-, -]$ , is a graded Lie algebra (see Def. 1.9).

All N-manifolds in this note arise from graded vector bundles<sup>4</sup>, as follows:

*Example 1.4.* Let  $F = \oplus_{i < 0} F_i \rightarrow M$  be a graded vector bundle. The N-manifold associated to it has body  $M$ , and  $\mathcal{O}_M$  is given by the sheaf of sections of  $S^\bullet F^*$ .

We will focus mainly on degree 1 N-manifolds, which we now describe in more detail. To do so we recall first

---

<sup>3</sup>Strictly speaking one should define vector fields in terms of the sheaf  $\mathcal{O}_M$  over  $M$ . However we will work only with objects defined on the whole of the body  $M$ , hence the above definition will suffice for our purposes.

<sup>4</sup>Actually it can be shown that every finite dimensional N-manifold is (non-canonically) isomorphic to one arising from a graded vector bundle.

**Definition 1.5.** Given a vector bundle  $E$  over  $M$ , a *covariant differential operator*<sup>5</sup> (CDO) is a linear map  $Y : \Gamma(E) \rightarrow \Gamma(E)$  such that there exists a vector field  $\underline{Y}$  on  $M$  (called *symbol*) with

$$Y(f \cdot e) = \underline{Y}(f)e + f \cdot Y(e), \quad \text{for } f \in C^\infty(M), e \in \Gamma(E). \quad (1)$$

We denote the set of CDOs on  $E$  by  $CDO(E)$ . If  $Y \in CDO(E)$ , then the dual  $Y^* \in CDO(E^*)$  is defined by

$$\langle Y^*(\xi), e \rangle + \langle \xi, Y(e) \rangle = \underline{Y}(\langle \xi, e \rangle), \quad \text{for all } e \in \Gamma(E), \xi \in \Gamma(E^*). \quad (2)$$

Recall that if  $E \rightarrow M$  is a (ordinary) vector bundle,  $E[1]$  denotes the graded vector bundle whose fiber over  $x \in M$  is  $(E_x)[1]$  (a graded vector space concentrated in degree  $-1$ ).

**Lemma 1.6.** *If  $E \rightarrow M$  is a vector bundle, then  $\mathcal{M} := E[1]$  is a degree 1  $N$ -manifold with body  $M$ , and conversely all degree 1  $N$ -manifolds arise this way.*

*The algebra of functions  $C(\mathcal{M})$  is generated by*

$$C_0(\mathcal{M}) = C^\infty(M) \text{ and } C_1(\mathcal{M}) = \Gamma(E^*).$$

*The  $C(\mathcal{M})$ -module of vector fields is generated by elements in degrees  $-1$  and  $0$ . We have identifications*

$$\chi_{-1}(\mathcal{M}) = \Gamma(E) \text{ and } \chi_0(\mathcal{M}) = CDO(E^*)$$

*induced by the actions on functions. Further the map  $\chi_0(\mathcal{M}) \cong CDO(E)$  obtained dualizing CDOs is just  $X_0 \mapsto [X_0, \cdot]$  (using the identification  $\chi_{-1}(\mathcal{M}) = \Gamma(E)$ ).*

The proof of Lemma 1.6 is given in Appendix A.1.

*Remark 1.7.* Let us choose coordinates  $\{x_i\}$  on an open subset  $U \subset M$  and a frame  $\{e_\alpha\}$  of sections of  $E|_U$ . Let  $\{\xi^\alpha\}$  be the dual frame for  $E^*|_U$ , and assign degree 1 to its elements. Then  $\{x_i, \xi^\alpha\}$  form a set of coordinates for  $\mathcal{M} := E[1]$  (in particular they generate  $C(\mathcal{M})$  over  $U$ ). The coordinate expression of vector fields is as follows.  $\chi_{-1}(\mathcal{M})$  consist of elements of the form  $f_\alpha \frac{\partial}{\partial \xi^\alpha}$ , and  $\chi_0(\mathcal{M})$  of elements of the form  $g_i \frac{\partial}{\partial x_i} + f_{\alpha\beta} \xi^\alpha \frac{\partial}{\partial \xi^\beta}$ . Here  $f_\alpha, g_i, f_{\alpha\beta} \in C^\infty(M)$ , for  $i \leq \dim(M)$  and  $\alpha, \beta \leq rk(E)$ , and we adopt the Einstein summation convention.

### 1.1.2 NQ-manifolds and Lie algebroids

We will be interested in  $N$ -manifolds equipped with extra structure:

**Definition 1.8.** An *NQ-manifold* is an  $N$ -manifold  $\mathcal{M}$  equipped with a *homological vector field*, i.e. a degree 1 vector field  $Q$  such that  $[Q, Q] = 0$ .

To shorten notation, we call a degree  $n$  NQ-manifold a *NQ- $n$  manifold*.

Before considering  $\chi(\mathcal{M})$ , we recall the notion of differential graded Lie algebra (DGLA):

**Definition 1.9.** A *graded Lie algebra* consists of a graded vector space  $L = \bigoplus_{i \in \mathbb{Z}} L_i$  together with a bilinear bracket  $[\cdot, \cdot] : L \times L \rightarrow L$  such that, for all homogeneous  $a, b, c \in L$ :

---

<sup>5</sup>Also known as derivative endomorphism, see [13, §1].

- the bracket is degree-preserving:  $[L_i, L_j] \subset L_{i+j}$
- the bracket is graded skew-symmetric:  $[a, b] = -(-1)^{|a||b|}[b, a]$
- the adjoint action  $[a, \cdot]$  is a degree  $|a|$  derivation of the bracket (Jacobi identity):  
 $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$ .

A *differential graded Lie algebra* (DGLA)  $(L, [\cdot, \cdot], \delta)$  is a graded Lie algebra together with a linear  $\delta : L \rightarrow L$  such that

- $\delta$  is a degree 1 derivation of the bracket:  $\delta(L_i) \subset L_{i+1}$  and  $\delta[a, b] = [\delta a, b] + (-1)^{|a|}[a, \delta b]$
- $\delta^2 = 0$ .

**Lemma 1.10.** *For a NQ- $n$  manifold  $\mathcal{M}$ , the space of vector fields*

$$(\chi(\mathcal{M}), [Q, -], [-, -])$$

*is a negatively bounded DGLA with lowest degree  $-n$ .*

*Proof.* The fact that  $[Q, -]$  squares to zero follows from  $[Q, [Q, -]] = \frac{1}{2}[[Q, Q], -] = 0$ . The fact that  $[Q, -]$  is a degree 1 derivation of the Lie bracket follows from the Jacobi identity. So the above is a DGLA.

A vector field on  $\mathcal{M}$  has local expression  $\sum_i f_i \frac{\partial}{\partial y_i}$ , where  $f_i \in C(\mathcal{M})$  and  $y_i$ 's are local coordinates on  $\mathcal{M}$ . The degree of  $\frac{\partial}{\partial y_i}$  is  $-deg(y_i)$ . Since  $deg(y_i) \in \{0, \dots, n\}$  we are done.  $\square$

*Remark 1.11.* As a  $C(\mathcal{M})$ -module,  $\chi(\mathcal{M})$  is generated by its elements in degrees  $-n, \dots, 0$ . This suggests that the most important information is contained in following the truncated DGLA (a sub-DGLA of  $\chi(\mathcal{M})$ ):

$$\chi_{-n}(\mathcal{M}) \oplus \dots \oplus \chi_{-1}(\mathcal{M}) \oplus \chi_0^Q(\mathcal{M}), \quad (3)$$

where

$$\chi_0^Q(\mathcal{M}) := \{X \in \chi_0(\mathcal{M}) : [Q, X] = 0\}.$$

A well-known example of NQ-manifolds is given by Lie algebroids [17].

**Definition 1.12.** A *Lie algebroid*  $A$  over a manifold  $M$  is a vector bundle over  $M$ , such that the global sections of  $A$  form a Lie algebra with Lie bracket  $[\cdot, \cdot]_A$  and Leibniz rule holds

$$[X, fY]_A = f[X, Y]_A + \rho(X)(f)Y, \quad X, Y \in \Gamma(A), f \in C^\infty(M),$$

where  $\rho : A \rightarrow TM$  is a vector bundle morphism called the *anchor*.

The following is well known ([26], see also [12]):

**Lemma 1.13.** *NQ-1 manifolds are in bijective correspondence with Lie algebroids.*

We describe the correspondence using the derived bracket construction. By Lemma 1.6 there is a bijection between vector bundles and degree 1 N-manifolds. If  $A$  is a Lie algebroid, then the homological vector field is just the Lie algebroid differential acting on  $\Gamma(\wedge^\bullet A^*) = C(A[1])$ . Conversely, if  $(\mathcal{M} := A[1], Q_A)$  is an NQ-manifold, then the Lie algebroid structure on  $A$  can be recovered by the derived bracket construction [12, §4.3]: using the identification  $\chi_{-1}(\mathcal{M}) = \Gamma(A)$  recalled in Lemma 1.6, we define

$$[a, a']_A = [[Q_A, a], a'], \quad \rho(a)f = [[Q_A, a], f], \quad (4)$$

where  $a, a' \in \Gamma(A)$  and  $f \in C^\infty(M)$ .

In coordinates the correspondence is as follows. Choose coordinates  $x_\alpha$  on  $M$  and a frame of sections  $e_i$  of  $A$ , inducing (degree 1) coordinates  $\xi_i$  on the fibers of  $A[1]$ . Then

$$Q_A = \frac{1}{2} \xi^j \xi^i c_{ij}^k(x) \frac{\partial}{\partial \xi_k} + \rho_i^\alpha(x) \xi^i \frac{\partial}{\partial x_\alpha} \quad (5)$$

where  $[e_i, e_j]_A = c_{ij}^k(x) e_k$  and the anchor of  $e_i$  is  $\rho_i^\alpha(x) \frac{\partial}{\partial x_\alpha}$ .

Viewing Lie algebroids as NQ-manifolds proves to be very valuable. For example, the definition of *Lie algebroid morphism*  $A \rightarrow A'$  is quite involved, but in terms of NQ-manifolds it is simply a morphism of N-manifolds from  $A[1]$  to  $A'[1]$  (i.e., a morphism of graded commutative algebras  $C(A'[1]) \rightarrow C(A[1])$ ) which respects homological vector field. Similarly, the notion of double Lie algebroid is quite involved, but it simplifies once expressed in terms of homological vector fields (Def. 2.5).

## 1.2 Strict Lie 2-algebra actions on NQ-1 manifolds

In this subsection we define strict Lie 2-algebra actions on NQ-1 manifolds in §1.2.1. Then in §1.2.2 we study actions from an ordinary differential geometry view point, that is, we view NQ-1 manifolds as Lie algebroids and describe the action by ordinary differential geometry data.

### 1.2.1 Definition of strict Lie 2-algebra action

**Definition 1.14.** A *strict Lie 2-algebra* (in the sense of [2]) is a DGLA (see Def. 1.9) concentrated in degrees  $-1$  and  $0$ .

Concretely, a strict Lie 2-algebra can be described as follows. It is given by  $\mathfrak{h}[1] \oplus \mathfrak{g}$  (where  $\mathfrak{h}, \mathfrak{g}$  are ordinary vector space) together with a Lie algebra structure on  $\mathfrak{g}$ , a left  $\mathfrak{g}$ -module structure on  $\mathfrak{h}$  (both of which we denote by  $[\cdot, \cdot]$ , and a linear map  $\delta: \mathfrak{h} \rightarrow \mathfrak{g}$  satisfying  $\delta[v, w] = [v, \delta w]$  for all  $v \in \mathfrak{h}, w \in \mathfrak{g}$ .

**Definition 1.15.** A *strict action* of a strict Lie 2-algebra  $\mathfrak{h}[1] \oplus \mathfrak{g}$  on an NQ-1 manifold  $\mathcal{M}$  is a morphism of DGLAs

$$\mu: \mathfrak{h}[1] \oplus \mathfrak{g} \rightarrow \chi(\mathcal{M}),$$

i.e., a degree-preserving linear map preserving the differentials and Lie brackets.

The NQ-1 manifold  $\mathcal{M}$  is equal to  $A[1]$  for some Lie algebroid  $A$ , by Lemma 1.13. We spell out what Def. 1.15 means: we have maps

$$\begin{aligned}\mu|_{\mathfrak{h}[1]}: \mathfrak{h}[1] &\rightarrow \chi_{-1}(A[1]) \\ \mu|_{\mathfrak{g}}: \mathfrak{g} &\rightarrow \chi_0(A[1])\end{aligned}$$

such that

$$\mu(\delta w) = d_Q(\mu(w)) \quad \text{for all } w \in \mathfrak{h}[1], \quad (6)$$

$$0 = d_Q(\mu(v)) \quad \text{for all } v \in \mathfrak{g}, \quad (7)$$

$$\mu[v, w] = [\mu(v), \mu(w)] \quad \text{for all } v \in \mathfrak{g}, w \in \mathfrak{h}[1], \quad (8)$$

$$\mu[v_1, v_2] = [\mu(v_1), \mu(v_2)] \quad \text{for all } v_i \in \mathfrak{g}. \quad (9)$$

*Remark 1.16.* By eq. (6), the image of the action map  $\mu$  will be contained in the truncated DGLA  $\chi_{-1}(\mathcal{M}) \oplus \chi_0^Q(\mathcal{M})$  (see eq. (3)). Hence Lie 2-algebra actions on  $\mathcal{M}$  can be formulated using only the truncated DGLA.

### 1.2.2 DGLAs and Lie algebra crossed modules

Let  $\mathfrak{h}[1] \oplus \mathfrak{g}$  be a strict Lie 2-algebra. Recall that  $[w, w']_\delta := [\delta w, w']$  makes  $\mathfrak{h}$  into a Lie algebra, which we denote by  $\mathfrak{h}_\delta$ . Let  $A[1]$  be an NQ-1 manifold. Using the identifications  $\chi_{-1}(A[1]) = \Gamma(A)$  and  $\chi_0(A[1]) \cong CDO(A)$  given in Lemma 1.6, we obtain a characterization of strict actions (Def. 1.15) in terms of classical geometric objects.

**Lemma 1.17.** *Let  $\mu: \mathfrak{h}[1] \oplus \mathfrak{g} \rightarrow \chi(\mathcal{M})$  be a linear map. Then  $\mu$  is a morphism of DGLAs iff*

- $\mu|_{\mathfrak{g}}: \mathfrak{g} \rightarrow CDO(A)$  is an infinitesimal action of  $\mathfrak{g}$  on  $A$  by infinitesimal Lie algebroid automorphisms, with the property that  $\delta(w)$  acts as  $[\mu(w), \cdot]_A$  for all  $w \in \mathfrak{h}$
- $\mu|_{\mathfrak{h}}: \mathfrak{h}_\delta \rightarrow \Gamma(A)$  is a Lie algebra homomorphism which is equivariant w.r.t. the representation of  $\mathfrak{g}$  on  $\mathfrak{h}$  by  $v \mapsto [v, \cdot]$  and the representation  $\mu|_{\mathfrak{g}}$  of  $\mathfrak{g}$  on  $\Gamma(A)$ .

*Proof.* Assume that  $\mu$  is a strict action. (9) means that  $\mu|_{\mathfrak{g}}$  is an infinitesimal action of  $\mathfrak{g}$  on the vector bundle  $A$ , and (7) means that the action is by infinitesimal Lie algebroid automorphisms.

(6) means that  $\delta w$  acts by  $[\mu(w), \cdot]_A$  for all  $w \in \mathfrak{h}$ , by the derived bracket construction.

(8) means that  $\mu|_{\mathfrak{h}}$  is an equivariant map.

Further  $\mu|_{\mathfrak{h}}$  is a Lie algebra morphism:

$$\mu[w_1, w_2]_\delta = \mu[\delta w_1, w_2] = [\mu(\delta w_1), \mu(w_2)] = [[Q, \mu(w_1)], \mu(w_2)] = [\mu(w_1), \mu(w_2)]_A,$$

where the second equality holds by (8) and the third equality by (6).

The converse implication is obtained reversing the argument.  $\square$

The idea behind Lemma 1.17 is the concept of crossed module, which might be more familiar to the reader than strict Lie 2-algebras, even though they are equivalent concepts (see Prop 1.19).



**Definition 1.18.** A *crossed module of Lie algebras*  $(\mathfrak{h}, \mathfrak{g}, \delta, \alpha)$  consists of a Lie algebra morphism  $\delta : \mathfrak{h} \rightarrow \mathfrak{g}$  and an (left) action of  $\mathfrak{g}$  on  $\mathfrak{h}$  by derivations, i.e.  $\alpha : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ , such that

$$\delta(\alpha(v)(w)) = [v, \delta(w)], \quad \alpha(\delta(w))w' = [w, w'].$$

where  $v \in \mathfrak{g}$ ,  $w, w' \in \mathfrak{h}$ .

A classical result (see [2, Thm. 36]) is that

**Proposition 1.19.** *There is a one-to-one correspondence between strict Lie 2-algebras and crossed modules of Lie algebra.*

Since a similar construction will show up again in §2.2, we recall the correspondence: A strict Lie 2-algebra  $(L_{-1}[1] \oplus L_0, \delta, [\cdot, \cdot])$  gives rise to a Lie algebra crossed module with  $\mathfrak{h} = L_{-1}$  and  $\mathfrak{g} = L_0$  where

$$\begin{aligned} [w, w']_{\mathfrak{h}} &:= [\delta(w), w'], \\ \alpha(v)(w) &:= [v, w], \\ [v, v']_{\mathfrak{g}} &:= [v, v'] \end{aligned}$$

for  $v, v' \in \mathfrak{g}$ ,  $w, w' \in \mathfrak{h}$ . The Jacobi identity of  $[\cdot, \cdot]$  gives the Jacobi identities of  $[\cdot, \cdot]_{\mathfrak{h}}$  and  $[\cdot, \cdot]_{\mathfrak{g}}$  and the remaining conditions for crossed modules.

On the other hand, a crossed module  $(\mathfrak{h}, \mathfrak{g}, \delta, \alpha)$  gives rise to a strict Lie-2 algebra with  $L_{-1} = \mathfrak{h}$ ,  $L_0 = \mathfrak{g}$ ,  $\delta$  as a differential, and

$$\begin{aligned} [w, w'] &:= 0, \\ [v, v'] &:= [v, v']_{\mathfrak{g}}, \\ [v, w] &= -[w, v] := \alpha(v)(w) \end{aligned}$$

for  $v, v' \in L_0$ ,  $w, w' \in L_{-1}$ . The Jacobi identity of  $[\cdot, \cdot]$  is implied by the Jacobi identities of  $[\cdot, \cdot]_{\mathfrak{h}}$  and  $[\cdot, \cdot]_{\mathfrak{g}}$  and various conditions for crossed modules.

Consequently, there is a 1-1 correspondence between DGLA morphisms and crossed module morphisms. Thus Lemma 1.17 basically explicitly tells us that given  $\mu : \mathfrak{h}[1] \oplus \mathfrak{g} \rightarrow \chi(\mathcal{M})$  a linear map, then  $\mu$  is a morphism of DGLAs iff  $(\mu|_{\mathfrak{h}}, \mu|_{\mathfrak{g}})$  is a morphism of Lie algebra crossed modules from the crossed module associated to  $\mathfrak{h}[1] \oplus \mathfrak{g}$  to the one associated to the truncated DGLA  $\chi_{-1}(A[1]) \oplus \chi_0(A[1])^Q$ . For this, we only need to notice that the crossed module associated to  $\chi_{-1}(A[1]) \oplus \chi_0(A[1])^Q$  by Lemma 1.19 is the quadruple given by  $(\Gamma(A), [\cdot, \cdot]_A)$ , the subset of  $CDO(A)$  consisting of infinitesimal Lie algebroid automorphisms, the morphism  $\delta(a) = [a, \cdot]_A$  and the natural action of  $CDO(A)$  on  $\Gamma(A)$ .

## 2 Integration to global actions

The purpose of this section is to integrate the infinitesimal actions introduced in Def. 1.15. We do so using the framework of Mackenzie's doubles, which we review in §2.1. In §2.2 we display the objects integrating strict Lie 2-algebras, and in §2.3 – the heart of this paper – we integrate the corresponding strict actions.

Recall that one can differentiate a Lie groupoid  $G_1 \rightrightarrows G_0$  to obtain its Lie algebroid. This defines a functor from the category of Lie groupoids to the category of Lie algebroids, called Lie functor. We refer to the inverse process as “integration”.

## 2.1 Background on Mackenzie's doubles

In this subsection we recall the formalism of Mackenzie doubles and its extension by Mehta. We will use it in §2.2 and §2.3 to integrate strict Lie 2-algebras and their actions.

Recall that one can apply the Lie functor to any Lie groupoid to obtain its Lie algebroid. Further, given a Lie algebroid, applying the degree shifting functor  $[1]$  one obtains an NQ-1 manifold (Lemma 1.13). The formalism of Mackenzie doubles relates *double Lie groupoids* (see Def. 2.2) to three other structures obtained applying (horizontally or vertically) the Lie functor. This was extended by Mehta [19] who applied (horizontally or vertically) the degree shifting functor  $[1]$  to obtain NQ-manifolds with additional structures. The situation is summarized in the following diagram taken from [19]:

$$\begin{array}{ccccc}
 \text{Double Lie groupoids} & \xrightarrow{\text{Lie}_H} & \mathcal{LA}\text{-groupoids} & \xrightarrow{[1]} & Q\text{-groupoids} \\
 \downarrow \text{Lie}_V & & \downarrow \text{Lie} & & \downarrow \text{Lie} \\
 \mathcal{LA}\text{-groupoids} & \xrightarrow{\text{Lie}} & \text{Double Lie algebroids} & \xrightarrow{[1]_H} & Q\text{-algebroids} \\
 \downarrow [1] & & \downarrow [1]_V & & \downarrow [1] \\
 Q\text{-groupoids} & \xrightarrow{\text{Lie}} & Q\text{-algebroids} & \xrightarrow{[1]} & \text{Double } Q\text{-manifolds}
 \end{array} \tag{10}$$

*Remark 2.1.* In general it is not known whether the Lie functors appearing in the above diagram can be inverted. For instance, given a double Lie algebroid whose vertical Lie algebroids are integrable to Lie groupoids, it is not known if the integrating Lie groupoids form an  $\mathcal{LA}$ -groupoid. The following question is also open: does a  $\mathcal{LA}$ -groupoid for which the Lie algebroid structures are integrable arise from a double groupoid? Partial answers to this problem were worked out in [25].

The portion of diagram (10) which is relevant to us is the following:

$$\begin{array}{ccc}
 \text{Double Lie groupoids} & \xrightarrow{\text{Lie}_H} & \mathcal{LA}\text{-groupoids} \\
 & & \downarrow \text{Lie} \\
 & & \text{Double Lie algebroids} \xrightarrow{[1]_H} Q\text{-algebroids}
 \end{array} \tag{11}$$

We define the objects appearing in it. We point out that Mehta [19] works entirely in the category of graded manifolds, whereas we want to assume that the double Lie groupoids appearing in (11) consist of ordinary manifolds. This explain why our definitions below are more restrictive than those of [19].

**Definition 2.2.** Let  $\mathbf{StrLgd}$  be the category of Lie groupoids with strict morphisms<sup>6</sup>. A *double Lie groupoid* is a groupoid object in  $\mathbf{StrLgd}$ . A *strict Lie 2-group* is a group object in  $\mathbf{StrLgd}$ . A *strict Lie 2-group action* is a group action in  $\mathbf{StrLgd}$ .

**Definition 2.3.** Let  $\mathbf{LA}$  be the category of Lie algebroid and Lie algebroid morphisms. A  $\mathcal{LA}$ -groupoid is a groupoid object in  $\mathbf{LA}$ . An  $\mathcal{LA}$ -group is a group object in  $\mathbf{LA}$ . An  $\mathcal{LA}$ -group action on a Lie algebroid is a group action in  $\mathbf{LA}$ .

<sup>6</sup>That is, usual Lie groupoid morphisms. We will not make use of the notion of generalized morphism, i.e. Hilsum-Skandalis bimodule.

In plain English, an  $\mathcal{LA}$ -group is a Lie algebroid  $C$  endowed with an additional group structure. For example, there is a multiplication

$$m : C \times C \rightarrow C$$

which is a Lie algebroid morphism and satisfies the (strict) associativity diagram. There are also an identity morphism and an inverse morphism

$$e : pt \rightarrow C, \quad i : C \rightarrow C,$$

which satisfy the group axioms. If  $C$  is a Lie algebroid over  $N$ , then these axioms say exactly that both  $C$  and  $N$  are Lie groups, and  $C \rightarrow N$  is a group morphism, i.e.  $N$  has an induced group structure from  $C$ .

*Remark 2.4.* A Q-groupoid is a groupoid object in the category of NQ-1 manifolds. We will not make use of this notion. By the correspondence between Lie algebroids and NQ-1 manifolds (see Lemma 1.13) it is clear that  $\mathcal{LA}$ -group(oid)s correspond to Q-group(oid)s.

**Definition 2.5.** A *double Lie algebroid* [15] is a double vector bundle

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \quad (12)$$

such that both vertical sides and both horizontal sides are Lie algebroids, subject to certain compatibility conditions (see for instance [28, §1]). By [28, Thm. 1], the compatibility conditions are equivalent to the following condition. If we apply the  $[1]$ -functor to the vertical sides, to obtain a vector bundle of graded manifolds  $D[1]_A \rightarrow B[1]$ , and then we apply again the  $[1]$ -functor to it, the resulting degree 2 graded manifold  $(D[1]_A)[1]$  will be endowed with homological vector fields encoding the Lie algebroid structures on  $D \rightarrow A$  and  $D \rightarrow B$ ; the condition is that these two vector fields commute.

**Definition 2.6.** A *Q-algebroid* [20, Def. 4.22] is a  $\mathbf{N1}$ -algebroid<sup>7</sup> [20, Def. 4.1]  $\mathcal{A} \rightarrow \mathcal{M}$  with a homological vector field  $Q$  which is *morphic* [20, Def. 4.14]. A Q-algebroid for which  $\mathcal{M}$  is a point is called a *Q-algebra*.

Q-algebras are exactly the same thing as strict Lie 2-algebras, as we will show in Lemma 2.8.

Definition 2.6 needs some explanation. An  $\mathbf{N1}$ -vector bundle<sup>8</sup> [20, Def. 2.1]  $\mathcal{E} \rightarrow \mathcal{M}$  consists of two degree 1 N-manifolds and a surjection between them, subject to a trivialization condition. A section [20, Def. 2.4] is just a map of N-manifolds  $s : \mathcal{M} \rightarrow \mathcal{E}$  (not necessarily degree-preserving) which composed with the projection equals  $Id_{\mathcal{M}}$ . A  $\mathbf{N1}$ -vector bundle  $\mathcal{E} \rightarrow \mathcal{M}$ , endowed with a (degree preserving) morphism from  $\Gamma(\mathcal{E}) \rightarrow \chi(\mathcal{M})$  and bracket  $\Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$  satisfying the graded Leibniz rule and Jacobi identity, forms a  $\mathbf{N1}$ -algebroid [20, Def. 4.1]. An example [19, Ex. 2.4.4] is given by the action algebroid

<sup>7</sup>Mehta refers to it as “superalgebroid” and allows for  $\mathbb{Z}$ -graded manifolds.

<sup>8</sup>Mehta [20] refers to it as “super vector bundle” and allows for arbitrary  $\mathbb{Z}$ -graded manifolds. In our notation, the prefix “ $\mathbf{N1}$ ” refers to the category of degree 1 N-manifolds.

induced from a morphism of graded Lie algebras into  $\chi(\mathcal{M})$ , for  $\mathcal{M}$  a degree 1 N-manifold. Graded Lie algebras concentrated in degrees  $-1$  and  $0$  are exactly the **N1**-algebroids for which the base  $\mathcal{M}$  is a point.

A vector field on a **N1**-algebroid  $\mathcal{A} \rightarrow \mathcal{M}$  is *morphic* [20, Def. 4.14] if it is linear in on the fibers (in the sense that its action on a function linear on the fibers is linear again) and, viewed as a vector field on  $\mathcal{A}[1]$ , it commutes with  $Q_{\mathcal{A}}$ . Here  $Q_{\mathcal{A}}$  is the homological vector field on  $\mathcal{A}[1]$  which, by virtue of [20, Thm. 4.6], encodes the **N1**-algebroid structure on  $\mathcal{A} \rightarrow \mathcal{M}$ .

## 2.2 Integrating strict Lie 2-algebras: $\mathcal{LA}$ -groups and 2-groups

In this subsection we describe the objects that integrate a strict Lie-2 algebra (Def. 1.14), namely strict 2-groups (Def. 2.2) and  $\mathcal{LA}$ -groups (Def. 2.3). The idea is

- §2.2.1: show that strict Lie-2 algebras fit in the framework of diagram (11)
- §2.2.2: chase back in diagram (11) to obtain the integrating objects.

### 2.2.1 What is an integration of a strict Lie 2-algebra?

An integration of a strict Lie 2-algebra  $\mathfrak{h}[1] \oplus \mathfrak{g}$  is *defined* to be a strict Lie 2-group [3] whose corresponding crossed module differentiates to the crossed module corresponding to  $\mathfrak{h}[1] \oplus \mathfrak{g}$  (Lemma 1.19).

Here we provide an integration procedure by viewing a strict Lie 2-algebras as a  $Q$ -algebroid<sup>9</sup> (Lemma 2.8). This point of view has the advantage of providing a hint for what the right notion of integration of a strict action of a strict Lie 2-algebra should be (see Def. 1.15).

Given a finite dimensional  $\mathbb{Z}$ -graded vector space  $L$ , a DGLA structure on  $L$  can be encoded by means of a homological vector field  $Q$  on  $L[1]$  (see Def. 1.8). We recall the procedure to recover the DGLA structure on  $L$  from  $Q$ ; it is a special case of Voronov's *higher derived brackets* construction [27, Ex. 4.1]. We have

$$\delta v = [Q, \iota_v]|_0 \quad \text{and} \quad [v_1, v_2]_L = (-1)^{|v_1|} [[Q, \iota_{v_1}], \iota_{v_2}]|_0$$

where  $|v|$  is the degree of  $v \in L$ ,  $\iota_v$  is the induced constant vector field on  $L[1]$ , and “ $|_0$ ” denotes the evaluation at the origin in  $L[1]$ .

*Remark 2.7.* To make things more explicit, we express the above construction in coordinates. If  $\{e_i\}$  is a basis of  $L$  and  $\xi^i$  are the corresponding coordinates on  $L[1]$ , then  $Q$  is at most quadratic:

$$Q = Q_{\delta} + Q_{br} = \left( \xi^i Q_i^k + \frac{1}{2} \xi^j \xi^i Q_{ij}^k \right) \frac{\partial}{\partial \xi^k} \quad (13)$$

where we denote by  $Q_{\delta}$  and  $Q_{br}$  respectively the linear and quadratic component of  $Q$ . One has

$$\delta e_i = (-1)^{|e_i|} Q_i^k e_k \quad \text{and} \quad [e_i, e_j]_L = (-1)^{|e_j|} Q_{ij}^k e_k. \quad (14)$$

---

<sup>9</sup>Equivalently we could view strict Lie 2-algebras as double  $Q$ -manifolds or as double Lie algebroids.

**Lemma 2.8.** *There is a bijection between strict Lie 2-algebras and Q-algebras, given by  $(L, [\cdot, \cdot]_L, \delta) \mapsto (L, [\cdot, \cdot]_L, -Q_\delta)$ .*

*Proof.* Recall from Def. 2.6 that a Q-algebra is a graded Lie algebra concentrated in degrees  $-1$  and  $0$ , together with a morphic vector field.

Let  $(L, [\cdot, \cdot]_L, \delta)$  be a strict Lie 2-algebra. All three of  $Q_{br}$ ,  $Q_\delta$  and  $Q_{br} + Q_\delta$  are self-commuting, because by the derived bracket construction they define  $L_\infty$ -structures on  $L$  (namely the graded Lie bracket  $[\cdot, \cdot]_L$ , the differential  $\delta$ , and the DGLA structure). In particular  $Q_\delta$  is a linear homological vector field on  $L[1]$ . Further

$$[Q_{br} + Q_\delta, Q_{br} + Q_\delta] = [Q_{br}, Q_{br}] + [Q_\delta, Q_\delta] + 2[Q_{br}, Q_\delta]$$

implies that  $[Q_{br}, Q_\delta] = 0$ , that is,  $Q_\delta$  is a morphic vector field. Clearly  $-Q_\delta$  is also a morphic vector field, so  $(L, [\cdot, \cdot]_L, -Q_\delta)$  is a Q-algebra. Reversing the argument we see that all Q-algebras arise this way.  $\square$

### 2.2.2 Integrating the strict Lie 2-algebra

Let  $\mathfrak{h}[1] \oplus \mathfrak{g}$  be a strict Lie 2-algebra. By Lemma 2.8 we can view it as a Q-algebra. In this subsection we argue that it lies in the image of the functors appearing in the diagram (11), as follows<sup>10</sup>:

$$\begin{array}{ccc} (H \rtimes G) \rightrightarrows G & \xrightarrow{\text{Lie}_H} & (\mathfrak{h} \rtimes G) \rightarrow G \\ & \downarrow \text{Lie} & \\ & (\mathfrak{h} \rtimes \mathfrak{g}) \rightarrow \mathfrak{g} & \xrightarrow{[1]_H} (\mathfrak{h}[1] \rtimes \mathfrak{g}, -Q_\delta). \end{array} \quad (15)$$

Recall that the integration of the strict Lie 2-algebra  $\mathfrak{h}[1] \oplus \mathfrak{g}$  is by definition the strict Lie 2-group in the upper left corner.

We describe the structures appearing in diagram (15), in particular the strict Lie 2-group in the upper left corner. The strict Lie 2-algebra  $\mathfrak{h}[1] \oplus \mathfrak{g}$  corresponds to the crossed module of Lie algebras  $(\mathfrak{g}, \mathfrak{h}_\delta, \delta, \alpha)$  (Lemma 1.19), where  $\mathfrak{h}_\delta$  denotes the Lie algebra structure on the vector space  $\mathfrak{h}$  with bracket  $[w_1, w_2]_\delta := [\delta w_1, w_2]$  and  $\alpha(v) = [v, \cdot]$  for  $v \in \mathfrak{g}$ . Consider the quadruple<sup>11</sup>  $(G, H, \mathbf{t}, \phi)$ . Here  $H$  and  $G$  are the simply connected Lie groups integrating  $\mathfrak{h}_\delta$  and  $\mathfrak{g}$ , the map  $\mathbf{t}: H \rightarrow G$  is the Lie group morphism integrating  $\delta$ , and the left action  $\phi$  of  $G$  on  $H$  (by automorphisms of  $H$ ) is obtained integrating the infinitesimal action (by Lie algebra derivations)  $\alpha$ .

- **Strict Lie 2-group**  $(H \rtimes G) \rightrightarrows G$ : The Lie 2-group<sup>12</sup> is as follows. Its Lie groupoid structure is the action groupoid of the  $H$  action on  $G$  via  $hg := \mathbf{t}(h) \cdot g$  (so the target of  $(g, h)$  is given by  $\mathbf{t}(h) \cdot g$  and its source by  $g$ ). The group structure on  $H \times G$  is

<sup>10</sup>The notation is chosen so to describe the *vertical* (Lie group or Lie algebra) structures. The bottom horizontal sides of the double Lie groupoid,  $\mathcal{LA}$ -groupoid, etc appearing in diagram (15) are just points, so they are omitted.

<sup>11</sup>This quadruple forms what is known as the *crossed module of Lie groups* integrating the above crossed module of Lie algebras.

<sup>12</sup>It is the Lie 2-group associated to a crossed module of Lie groups [3].

the semidirect product structure by the action  $\phi$ . Explicitly, and using the notation  $G_\bullet = (G_1 \rightrightarrows G_0)$  for the Lie 2-group, the group structure is given by

$$m : G_1 \times G_1 \rightarrow G_1, \quad m((h_1, g_1), (h_2, g_2)) = (h_1 \cdot \phi(g_1)(h_2), g_1 g_2)$$

over the base map  $m(g_1, g_2) = g_1 g_2$ .

- **$\mathcal{LA}$ -group  $(\mathfrak{h} \rtimes G) \rightarrow G$ :** Its Lie algebroid structure is obtained by differentiating the Lie groupoid structure of the strict Lie 2-group  $G_\bullet$ , hence it is the transformation algebroid of the infinitesimal action of  $\mathfrak{h}_\delta$  on  $G$  by  $w \mapsto \delta w$  (the right-invariant vector field whose value at the identity is  $\delta w$ ).

The group multiplication on  $\mathfrak{h} \times G$  is the Lie algebroid morphism corresponding to  $m : G_1 \times G_1 \rightarrow G_1$ , i.e.,  $(w_1, g_1)(w_2, g_2) = (w_1 + g_1 w_2, g_1 g_2)$ . In other words, the group structure on  $\mathfrak{h} \times G$  is the semidirect product by the action of  $G$  on the vector space  $\mathfrak{h}$  obtained integrating  $\alpha$ .

- **Double Lie algebroid  $(\mathfrak{h} \rtimes \mathfrak{g}) \rightarrow \mathfrak{g}$ :** Notice first that applying  $Lie_V$  to the strict Lie 2-group above one obtains the Lie groupoid in the category of Lie algebras  $(\mathfrak{h}_\delta \rtimes \mathfrak{g}) \rightrightarrows \mathfrak{g}$ . The Lie groupoid structure is the transformation groupoid of the action of the abelian Lie algebra  $\mathfrak{h}$  on  $\mathfrak{g}$  which sends  $w \in \mathfrak{h}$  to the translation by  $\delta w$ .

Differentiating this Lie groupoid structure we obtain the Lie algebroid structure of our double Lie algebroid, which therefore<sup>13</sup> is the transformation algebroid of the infinitesimal action of the abelian Lie algebra  $\mathfrak{h}$  on  $\mathfrak{g}$  which sends  $w$  to the constant vector field  $\delta w$ .

The Lie algebra structure of our double Lie algebroid is obtained differentiating the Lie group structure of the  $\mathcal{LA}$ -group, so it is the semidirect product  $\mathfrak{h} \rtimes \mathfrak{g}$  of the action  $\alpha$  of  $\mathfrak{g}$  on the vector space (abelian Lie algebra)  $\mathfrak{h}$ . Explicitly:  $[(w_1, v_1), (w_2, v_2)] = ([v_1, w_2] - [v_2, w_1], [v_1, v_2])$ .

- **Q-algebra  $(\mathfrak{h}[1] \oplus \mathfrak{g}, -Q_\delta)$ :** Applying the functor  $[1]_H$  to the double Lie algebroid we obtain the NQ-manifold  $(\mathfrak{h}[1] \oplus \mathfrak{g}, -Q_\delta)$ . (To see this, recall that the anchor of a transformation Lie algebroid is given by the corresponding infinitesimal action, and use eq. (5) and eq. (14).) It has the graded Lie algebra structure  $\mathfrak{h}[1] \rtimes \mathfrak{g}$ .

The latter  $Q$ -algebra structure is exactly the one corresponding to our original strict Lie-2 algebra by Lemma 2.8. Hence we obtain:

**Lemma 2.9.** *The integration of the strict Lie-2 algebra  $\mathfrak{h}[1] \oplus \mathfrak{g}$  is given by the strict Lie 2-group  $(H \rtimes G) \rightrightarrows G$  described above.*

*Remark 2.10.* As mentioned above, a strict Lie 2-algebra can be viewed as a  $Q$ -algebra, or equivalently as a double Lie algebroid, and the integration we perform in §2.2.2 is its integration to a strict Lie 2-group.

<sup>13</sup>Here we are using [16, Thm 2.3], which states that starting from a double Lie groupoid and applying the functors  $Lie_H \circ Lie_V$  or  $Lie_V \circ Lie_H$ , one obtains the same double algebroid *up to* a canonical isomorphism. The canonical isomorphism in our case is the identity on  $\mathfrak{h} \times \mathfrak{g}$ . This follows from a simple computation (see the proof of [16, Thm 2.3]) representing each element  $(w, v) \in \mathfrak{h} \times \mathfrak{g}$  as second derivative of the map  $\gamma : \mathbb{R}^2 \rightarrow H \times G, \gamma(t, u) = (\exp_{\mathfrak{h}}(tuw), \exp_{\mathfrak{g}}(tv))$ .

Another instance of Lie 2-algebroid that fits into the framework of double Lie algebroids is given by a Courant algebroid  $A \oplus A^*$  arising from a Lie bialgebroid  $(A, A^*)$ . In this case, however, the integration of the corresponding double Lie algebroid via diagram (11) does not coincide with the integration of the Courant algebroid  $A \oplus A^*$ . The relation between the two integrations is not yet clear to us.

### 2.3 Integrating strict actions: $\mathcal{LA}$ -group actions on Lie algebroids and 2-group actions on Lie groupoids

This subsection is the heart of the paper: we define the notion of global action integrating a strict action  $\mu$  as in Def. 1.15, and show that the global action exists. The idea is

- §2.3.1: show that the transformation algebroid of the action  $\mu$  fits in the framework of diagram (11)
- §2.3.2: chase back in the diagram to obtain certain transformation groupoids, and describe the corresponding actions.

The diagram of Mackenzie's doubles (11), in the setup at hand, is displayed in (19) (it extends our previous diagram (15)). The various actions involved are displayed just before Thm. 2.16.

We end this subsection with remarks on the integrated actions (§2.3.3).

As earlier, we let  $\mathfrak{h}[1] \oplus \mathfrak{g}$  be a strict Lie-2 algebra and  $A \rightarrow M$  a Lie algebroid (so  $A[1]$  is a NQ-1 manifold).

#### 2.3.1 What is an integration of a strict Lie-2 algebra action?

The following proposition, which is an analogue to Lemma 2.8, associates a Q-algebroid to the strict action  $\mu$ .

**Proposition 2.11.** *Let  $\mathfrak{h}[1] \oplus \mathfrak{g}$  be a strict Lie-2 algebra,  $A$  be a Lie algebroid. We consider a morphism of graded Lie algebras  $\mu: \mathfrak{h}[1] \oplus \mathfrak{g} \rightarrow \chi(A[1])$ . Then  $\mu$  is a morphism of DGLAs (i.e. it respects differentials) iff the transformation algebroid of the action  $\mu$*

$$\begin{array}{c} T_\mu := (\mathfrak{h}[1] \oplus \mathfrak{g})[1] \times A[1] \\ \downarrow \\ A[1], \end{array} \quad (16)$$

*endowed with the homological vector field  $-Q_\delta + Q_A$ , is a Q-algebroid. Recall that  $Q_\delta$  was defined in Remark 2.7, and  $Q_A$  encodes the Lie algebroid structure on  $A$  as in Lemma 1.13.*

*Proof.* Denote  $L := \mathfrak{h}[1] \oplus \mathfrak{g}$ . The vector field  $-Q_\delta + Q_A$  is homological and it is linear in the fibers. So, in order to show that it is morphic, we just have to show that, once we view it as a vector field on  $L[1] \times A[1]$ , it commutes with the homological vector field encoding the Lie algebroid structure of (16). The latter is  $Q_{br} + Q_{action}$ , where  $Q_{br}$  was defined in Remark 2.7 and  $Q_{action}$  is defined in eq. (18) below and encodes the action  $\mu$ .

To show

$$[-Q_\delta + Q_A, Q_{br} + Q_{action}] = 0. \quad (17)$$



we proceed as follows.

Take bases  $\{v_i\}$  of  $\mathfrak{g}$  and  $\{w_\alpha\}$  of  $\mathfrak{h}[1]$ , whose dual bases induce coordinates  $\eta^i$  on  $\mathfrak{g}[1]$  and  $P^\alpha$  of  $\mathfrak{h}[2]$  (of degrees 1 and 2 respectively).

If the differential on  $L$  is given by  $\delta w_\alpha = D_{\alpha i} v_i$ , then in coordinates

$$Q_\delta = -P^\alpha D_{\alpha k} \frac{\partial}{\partial \eta^k}.$$

Further

$$Q_{action} := \eta^i \mu(v_i) - P^\alpha \mu(w_\alpha) \quad (18)$$

encodes the  $L$ -action on  $A[1]$

We have  $[Q_A, Q_{br}] = 0$ , since the two vector fields are defined on different manifolds, and  $[-Q_\delta, Q_{br}] = 0$  as shown in Lemma 2.8. Since

$$[Q_A, Q_{action}] = -\eta^i [Q_A, \mu(v_i)] - P^\alpha [Q_A, \mu(w_\alpha)]$$

and

$$[-Q_\delta, Q_{action}] = P^\alpha D_{\alpha k} \mu(v_k) = P^\alpha \mu(\delta w_\alpha)$$

we conclude that (17) holds iff  $[Q_A, \mu(w_\alpha)] = \mu(\delta w_\alpha)$  for all  $w_\alpha$  and  $[Q_A, \mu(v_i)] = 0$  for all  $v_i$ , which means that  $\mu$  respects differentials.  $\square$

*Remark 2.12.* 1) Prop. 2.11 together with diagram (10) imply that

$$((\mathfrak{h}[1] \times \mathfrak{g})[1] \times A[1], Q_\delta + Q_A, Q_{br} + Q_{action})$$

is a double  $Q$ -manifold [28]. When the action  $\mu$  is not necessarily strict (in the sense of [30]), the above is no longer a double  $Q$ -manifold. However in that case one can show [21] that the sum of the four above vector fields is still a homological vector field. This makes (16) into an “action” Lie 2-algebroid. Thus the integration of the action should be encoded by the Lie 2-groupoid integrating this action Lie 2-algebroid.

In the strict case, we will see below that the integration of the action is given by a double Lie groupoid ( $T_\Phi$  in diagram (19)). Moreover in this case, unlike the case of Courant algebroids (see Remark 2.10), there is a Lie 2-groupoid, obtained applying the Artin-Mazur’s codiagonal construction [1] (see Remark 2.19) to  $T_\Phi$ , which is also to be considered an integration of the action. These two integrating objects are not exactly equivalent: the double Lie groupoid contains more information because the double Lie algebroid contains more information than the Lie 2-algebroid.

2) [20, Thm. 6.2], which is proved without the explicit use of coordinates, is a special case of Prop. 2.11 (namely, the special case we consider in Ex. 3.1).

Now assume that

$$\boxed{\mu: \mathfrak{h}[1] \oplus \mathfrak{g} \rightarrow \chi(A[1])}$$

is a strict action (Def. 1.15), i.e. a morphism of DGLAs. Prop. 2.11 allows to view the action  $\mu$  in the framework of diagram (11).

We make the following definitions of integration of  $\mu$ .



**Definition 2.13.** The *integration* of the strict action  $\mu$  above is either of the following actions:

- I) An  $\mathcal{LA}$ -group action  $\Psi$  of  $(\mathfrak{h} \rtimes G) \rightarrow G$  on  $A \rightarrow M$  such that applying the functor  $\text{Lie}_V$  to its transformation groupoid one obtains the Q-algebroid (16).
- II) A strict Lie 2-group action  $\Phi$  of  $(H \rtimes G) \rightrightarrows G$  on  $\Gamma \rightrightarrows M$  such that applying the functors  $\text{Lie}_H$  and  $\text{Lie}_V$  to its transformation groupoid one obtains the Q-algebroid (16). (Here  $\Gamma \rightrightarrows M$  denotes the source simply connected Lie groupoid integrating  $A$ .)

Def. 2.13 requires some explanation. For a small category  $\mathcal{C}$  with direct products, a group object  $G$  in  $\mathcal{C}$ , and an object  $N$  in  $\mathcal{C}$ , the transformation groupoid is  $G \times N \rightrightarrows N$  as for usual Lie group action, and it is a groupoid object in  $\mathcal{C}$  [25, Prop. 3.0.15]. The transformation groupoid appearing in I) is the one corresponding to the group action  $\Psi$ . As  $\Psi$  is a group action in the category  $\mathbf{LA}$ , the transformation groupoid is a groupoid in  $\mathbf{LA}$ , (an  $\mathcal{LA}$ -groupoid), and hence fits in diagram (11). The same reasoning, applied to the category  $\mathbf{StrLgd}$ , holds for II).

The situation is summarized in diagram (19) in §2.3.2, which is the diagram of Mackenzie's doubles (11) for the above transformation groupoids.

Applying the vertical integration functor to a double Lie algebroid whose vertical structures are *transformation* algebroids (such as the one associated to (16) above), one expects to obtain an  $\mathcal{LA}$ -groupoid whose vertical structures are *transformation* groupoids.

Unfortunately there are no general enough statements about the integration of double algebroids to  $\mathcal{LA}$ -groupoids or  $\mathcal{LA}$ -groupoid to double groupoids (see Remark 2.1), so we can not “chase back” in diagram (11), but rather in §2.3.2 we have to check explicitly that the expected fact mentioned above is true for the double algebroid (16).

Moreover our method has the advantage that it provides explicit formulae.

### 2.3.2 Integrating the strict action

Now we integrate the strict Lie 2-algebra action  $\mu$  on a Lie algebroid  $A$  to both a  $\mathcal{LA}$ -group action on  $A$  and a Lie 2-group action on the source-simply connected Lie groupoid of  $A$ .

By virtue of the identification of Remark A.1,  $\mu$  induces a Lie algebra morphism

$$\tilde{\mu}: \mathfrak{h} \rtimes \mathfrak{g} \rightarrow \chi(A).$$

Here  $\mathfrak{h} \rtimes \mathfrak{g}$  is the semidirect product of  $\mathfrak{g}$  and the abelian Lie algebra  $\mathfrak{h}$ , and agrees with the Lie algebra structure induced by the original graded Lie algebra structure on  $\mathfrak{h}[1] \oplus \mathfrak{g}$ . In other words, we have an infinitesimal action of the Lie algebra  $\mathfrak{h} \rtimes \mathfrak{g}$  on  $A$ . Using this identification,  $w \in \mathfrak{h}$  maps to  $\mu(w)$  seen as a constant vertical vector field on  $A$  and  $v \in \mathfrak{g}$  maps to the infinitesimal vector bundle automorphism given by  $\mu(v)$ .

Assume that the infinitesimal action  $\tilde{\mu}|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \chi(A)$  is *complete* in the sense that the image of  $\tilde{\mu}$  consists of complete vector fields.

Then the infinitesimal action  $\tilde{\mu}|_{\mathfrak{g}}$  can be integrated to a Lie group action

$$\psi: G \times A \rightarrow A$$

of  $G$  on  $A$ , where  $G$  denotes the simply connected Lie group integrating  $\mathfrak{g}$ . Notice that  $\psi$  acts by Lie algebroid automorphisms of  $A$ , as a consequence of Lemma 1.17.

**Proposition 2.14.** *Consider the Lie algebroids  $A \rightarrow M$  and  $\mathfrak{h} \rtimes G \rightarrow G$  (as in §2.2.2). The Lie group action on  $A$  obtained integrating the infinitesimal action  $\tilde{\mu}$  is*

$$\boxed{\begin{array}{ccc} \Psi: (\mathfrak{h} \rtimes G) \times A & \rightarrow & A \\ (w, g), a_x & \mapsto & \psi(g, a_x) + \mu(w)|_{gx} \end{array}}$$

where  $x \in M$  and  $a_x \in A_x$ .

Further  $\Psi$  is a Lie algebroid morphism. In other words,  $\Psi$  is an  $\mathcal{LA}$ -group action.

The proof of Prop. 2.14 is given in Appendix A.2.

Integrating the Lie algebroid  $\mathfrak{h} \rtimes G \rightarrow G$  we obtain the strict Lie 2-group  $(H \rtimes G) \rightrightarrows G$  described in §2.2.2, where  $H$  is simply connected. Assume that the Lie algebroid  $A \rightarrow M$  is integrable to a source simply connected Lie groupoid  $\Gamma$ .

**Theorem 2.15.** *Consider the Lie groupoids  $\Gamma \rightrightarrows M$  and the Lie 2-group  $(H \rtimes G) \rightrightarrows G$ . The Lie groupoid morphism*

$$\boxed{\Phi: (H \rtimes G) \times \Gamma \rightarrow \Gamma}$$

integrating the Lie algebroid morphism  $\Psi$  is a Lie 2-group action.

In Prop. 2.17 we will give a description of the action  $\Phi$ .

*Proof.* Notice that  $\Phi$  is a well-defined Lie groupoid morphism as its domain is source simply connected.

We show that  $\Phi$  is a group action. Recall from §2.2.2 that the group multiplication  $m: (H \rtimes G) \times (H \rtimes G) \rightarrow (H \rtimes G)$  is the Lie groupoid morphism which integrates the multiplication  $\tilde{m}$  on  $\mathfrak{h} \rtimes G$ . Hence integrating both sides of the equality of Lie algebroid morphisms

$$\Psi \circ (\tilde{m} \times Id_A) = \Psi \circ (Id_{\mathfrak{h} \rtimes G} \times \Psi),$$

which holds because  $\Psi$  is a group action, one obtains

$$\Phi \circ (m \times Id_\Gamma) = \Phi \circ (Id_{H \rtimes G} \times \Phi).$$

This shows that  $\Phi$  is a group action both at the level of objects and of morphisms, hence it is a Lie 2-group action.  $\square$

The various actions appearing in this subsection can be summarized in the following diagram:

$$\boxed{\begin{array}{ccc} & & \mu: \mathfrak{h}[1] \oplus \mathfrak{g} \rightarrow \chi(A[1]) \\ & \rightsquigarrow & \tilde{\mu}: \mathfrak{h} \rtimes \mathfrak{g} \rightarrow \chi(A) \\ \text{Integrate the Lie algebra action} & \rightsquigarrow & \Psi: (\mathfrak{h} \rtimes G) \times A \rightarrow A \\ \text{Integrate the Lie algebroid morphism} & \rightsquigarrow & \Phi: (H \rtimes G) \times \Gamma \rightarrow \Gamma \end{array}}$$

**Theorem 2.16.** *The actions  $\Phi$  and  $\Psi$  defined above are really integrations of  $\mu$  in the sense of Def. 2.13 I) and II) respectively.*

*Proof.* We consider the double Lie groupoid  $T_\Phi$  obtained by taking the transformation groupoid of the action  $\Phi$  and the  $\mathcal{LA}$ -groupoid  $T_\Psi$  obtained by taking the transformation groupoid of the action  $\Psi$ . Notice that  $T_\Phi$  is really a double Lie groupoid by Prop. [25, Prop. 3.0.15] together with Thm. 2.15, and  $T_\Psi$  is really an  $\mathcal{LA}$ -groupoid because of Prop. [25, Prop. 3.0.15] together with Prop. 2.14.

Further we consider the double algebroid  $T_{\tilde{\mu}}$  consisting of the transformation algebroid of the action  $\tilde{\mu}: \mathfrak{h} \ltimes \mathfrak{g} \rightarrow \chi(A)$  and of the  $\mathfrak{g}$  action on  $M$  obtained by restricting  $\tilde{\mu}$  (vertically), together with  $A \rightarrow M$  and the transformation algebroid of the action of the abelian Lie algebra  $\mathfrak{h}$  on  $\mathfrak{g}$  which sends  $w \in \mathfrak{h}$  to the constant vector field  $\delta w$  (vertically):

$$\begin{array}{ccc} T_{\tilde{\mu}} = (\mathfrak{h} \ltimes \mathfrak{g}) \times A & \longrightarrow & \mathfrak{g} \times M \\ \downarrow & & \downarrow \\ A & \longrightarrow & M. \end{array}$$

Applying to  $T_{\tilde{\mu}}$  the horizontal degree shifting functor  $[1]_H$  we obtain (16) together with the homological vector field  $-Q_\delta + Q_A$  (see §2.2.2). Since by Prop. 2.11 it is a  $Q$ -algebroid, from Voronov's work ([28, Thm. 1], see also [28, §3]) it follows in particular that  $T_{\tilde{\mu}}$  is a really a double Lie algebroid.

$T_\Phi, T_\Psi$  and  $T_{\tilde{\mu}}$  fit into the upper row of the following table.

$$\begin{array}{ccccc} T_\Phi = (H \ltimes G) \times \Gamma & \rightrightarrows & G \times M & & T_\Psi = (\mathfrak{h} \ltimes G) \times A \longrightarrow G \times M \\ \parallel & & \parallel & \xrightarrow{\text{Lie}_H} & \parallel \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \rightrightarrows & M & & A \longrightarrow M \\ & & & & \downarrow \text{Lie} \\ & & T_{\tilde{\mu}} = (\mathfrak{h} \ltimes \mathfrak{g}) \times A & \longrightarrow & \mathfrak{g} \times M \\ & & \downarrow & & \downarrow \\ & & A & \longrightarrow & M \\ & & & & \xrightarrow{[1]_H} \\ & & & & \downarrow \\ & & & & A[1] \end{array} \quad \begin{array}{c} T_\mu = (\mathfrak{h}[1] \ltimes \mathfrak{g}) \times A[1] \\ \downarrow \\ A[1] \end{array} \quad (19)$$

We check that applying the functor  $\text{Lie}_H$  to  $T_\Phi$  we obtain  $T_\Psi$ . Clearly the horizontal Lie algebroid structures obtained this way are  $A \rightarrow M$  and its product with the Lie algebroid structure of the  $\mathcal{LA}$ -group  $\mathfrak{h} \ltimes G \rightarrow G$  (see §2.2.2). The left vertical Lie groupoid structure is obtained applying  $\text{Lie}_H$  to the maps defining the vertical Lie algebroid structures in  $T_\Phi$ . It is the transformation groupoid for the action  $\Psi$ , since the Lie algebroid morphism  $\Psi$  is obtained differentiating the Lie groupoid morphism  $\Phi$ .

Next we check that applying the functor  $\text{Lie}_V$  to  $T_\Psi$  we obtain  $T_{\tilde{\mu}}$ . Since the vertical groupoids of  $T_\Psi$  are transformation groupoids for group actions (of  $\mathfrak{h} \ltimes G$  and  $G$  respectively),

applying the functor  $\text{Lie}_V$  we obtain the transformation algebroids of the corresponding infinitesimal actions, which by Prop. 2.14 are  $\tilde{\mu}$  and the  $\mathfrak{g}$ -action on  $M$  respectively. The horizontal Lie algebroids in  $T_\Psi$  are the Lie algebroid  $A \rightarrow M$  and its product with the Lie algebroid structure on the  $\mathcal{LA}$ -group  $\mathfrak{h} \rtimes G \rightarrow G$ . Since the application of the Lie functor to the vertical groupoids of  $T_\Psi$  does not affect their spaces of units ( $A$  and  $M$  respectively), as horizontal Lie algebroid structures we obtain again  $A \rightarrow M$  and its product with the Lie algebroid structure of the double Lie algebroid  $\mathfrak{h} \rtimes \mathfrak{g} \rightarrow \mathfrak{g}$  (see §2.2.2). Altogether we obtain exactly  $T_{\tilde{\mu}}$ .

Finally, we saw in the first part of this proof that applying to  $T_{\tilde{\mu}}$  the horizontal degree shifting functor  $[1]_H$  we obtain the  $Q$ -algebroid  $T_\mu$  given in (16).

From this we conclude that  $\Psi$  and  $\Phi$  are integrations of the action  $\mu$  in the sense of Def. 2.13 I) and II) respectively.  $\square$

It is abstract to define the Lie 2-group action  $\Phi$  in Thm. 2.15 as the integration of the strict action  $\mu$  we started with. Now we give an explicit description of  $\Phi$ .

**Proposition 2.17.** *The Lie 2-group action  $\Phi: (H \rtimes G) \times \Gamma \rightarrow \Gamma$  can be described in terms of  $\tilde{\mu}$  as follows:*

- a)  $g \in G$  acts by the Lie groupoid automorphism of  $\Gamma$  which integrates the Lie algebroid automorphism  $\psi(g, \cdot)$  of  $A$ , where  $\psi$  denotes the Lie group action of  $G$  on  $A$  obtained by integrating the Lie algebra action  $\tilde{\mu}|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \chi(A)$ .
- b) the Lie group action of  $H$  on  $\Gamma$  is obtained integrating the Lie algebra action

$$\mathfrak{h}_\delta \rightarrow \chi^{\text{right}}(\Gamma), w \mapsto \overrightarrow{\tilde{\mu}(w)},$$

where for any section  $s$  of  $A \cong (\ker \mathbf{s}_*)|_M$  we denote by  $\overrightarrow{s}$  its extension as a right invariant vector field on  $\Gamma$ .

Since a general element of  $H \rtimes G$  can be written as  $(h, g) = (h, e)(e, g)$ , this gives a complete description of the action  $\Phi$ .

*Proof.* a) Fix  $g \in G$ . Consider  $\Phi((e, g), \cdot): \Gamma \rightarrow \Gamma$ . Its derivative restricted to  $(\ker \mathbf{s}_*)|_M \cong A$  is  $\Psi((0, g), \cdot) = \psi(g, \cdot): A \rightarrow A$ , since  $\Phi$  is the Lie groupoid morphism integrating the Lie algebroid morphism  $\Psi$ .

b) Let  $w \in \mathfrak{h}$ , denote by  $W$  the corresponding vector field on  $\Gamma$  induced by the action  $\Phi$ , and let  $h(t) := \exp_{\mathfrak{h}_\delta}(tw) \in H$ . For any  $m \in M$  we have

$$W_m = \frac{d}{dt}\bigg|_0 \Phi((h(t), e), m) = \Psi((w, e), 0_m) = \tilde{\mu}(w)_m,$$

so  $W|_M = \tilde{\mu}(w)$ . We want to show that  $W$  is a right-invariant vector field, i.e., that if  $x, y \in \Gamma$  are composable elements then  $(R_y)_*W_x = W_{x \circ y}$ , where  $\circ$  denotes the groupoid composition on  $\Gamma$ . Since  $\Phi$  is a Lie groupoid morphism (Thm. 2.15) and the elements  $(h(t), e)$  and  $(e, e)$  of the groupoid  $H \rtimes G$  are composable, we have

$$\Phi(((h(t), e), x)) \circ \Phi((e, e), y) = \Phi((h(t), e), x \circ y),$$

and since  $\Phi((e, e), y) = y$  applying the time derivative at  $t = 0$  we obtain exactly  $(R_y)_*W_x = W_{x \circ y}$ . This shows that  $W = \overrightarrow{\tilde{\mu}(w)}$  and concludes the proof.  $\square$

*Remark 2.18.* Prop. 2.17 allows also to describe the Lie algebra action on  $\Gamma$  corresponding to the Lie group action  $\Phi$ . It maps  $v \in \mathfrak{g}$  to the multiplicative vector field on  $\Gamma$  which corresponds to the vector field  $\tilde{\mu}(v)$  on  $A$ . It maps  $w \in \mathfrak{h}$  to  $\overrightarrow{\tilde{\mu}(w)}$ . This shows that the action  $\Phi$  coincides with the action constructed in [8, §11.2] in the special case where  $A$  is the Lie algebroid of a Poisson manifold. Notice that the action of [8, §11.2] was constructed “integrating”  $T_{\tilde{\mu}}$  (see diagram in the proof of Thm. 2.16) first horizontally and then vertically, while we constructed  $\Phi$  “integrating”  $T_{\tilde{\mu}}$  in the opposite order.

### 2.3.3 Lie 2-groupoids and principality of the integrated action

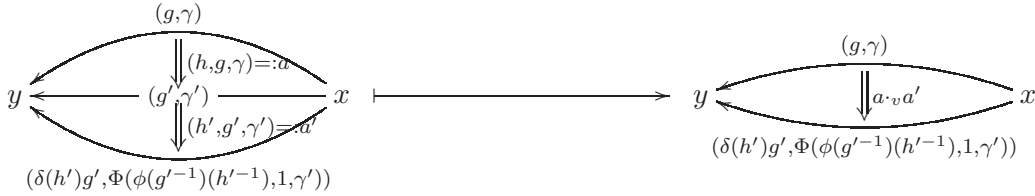
First, we make a remark on the integration of the Lie 2-algebroid (16). This is the point of view that should be generalized to integrate non-strict actions (in the sense of [30]).

*Remark 2.19.* We describe the strict Lie 2-groupoid integrating the Lie 2-algebroid (16), obtained by applying the Artin-Mazur’s codiagonal construction to the double groupoid  $T_{\Phi}$  appearing in (19). A strict Lie 2-groupoid is a groupoid object in the category of Lie groupoids whose space of objects (object-groupoid) is just a manifold. Our strict Lie 2-groupoid has as space of objects  $M$  (the base of  $A$ ), as object space of the morphism-groupoid (space of 1-arrows)  $G \times \Gamma$ , and as morphism space of the morphism-groupoid (space of 2-arrows)  $H \times G \times \Gamma$ . The space of 2-arrows is a Lie groupoid over the space of 1-arrows with source and target defined by

$$\mathbf{s}(h, g, \gamma) = (g, \gamma), \quad \mathbf{t}(h, g, \gamma) = (\delta(h)g, \Phi(\phi(g^{-1})(h^{-1}), 1, \gamma)),$$

and groupoid multiplication  $\cdot_v$  (vertical product) defined by

$$(h, g, \gamma) \cdot_v (h', g', \gamma') = (hh', g', \gamma\gamma').$$



Moreover, there are two maps  $\mathbf{t}_0, \mathbf{s}_0 : G \times \Gamma \rightarrow M$  defined by  $\mathbf{s}_0(g, \gamma) = \mathbf{s}_{\Gamma}(\gamma)$  and  $\mathbf{t}_0(g, \gamma) = \psi(g, \mathbf{t}_{\Gamma}(\gamma))$  via the source and target of  $\Gamma$  (denoted as  $x$  and  $y$  respectively in the above picture), so that we define the horizontal multiplication  $\cdot_h$

$$\begin{array}{ccc} (H \times G \times \Gamma) \times_{\mathbf{s}_0, M, \mathbf{t}_0} (H \times G \times \Gamma) & \xrightarrow{\cdot_h} & H \times G \times \Gamma, \\ \Downarrow & & \Downarrow \\ (G \times \Gamma) \times_{\mathbf{s}_0, M, \mathbf{t}_0} (G \times \Gamma) & \xrightarrow{\cdot_h} & G \times \Gamma \end{array}$$

by

$$\begin{aligned} (h_1, g_1, \gamma_1) \cdot_h (h_2, g_2, \gamma_2) &= (h_1 \phi(g_1)(h_2), g_1 g_2, \Phi(1, g_2^{-1}, \gamma_1) \gamma_2), \\ (g_1, \gamma_1) \cdot_h (g_2, \gamma_2) &= (g_1 g_2, \Phi(1, g_2^{-1}, \gamma_1) \gamma_2). \end{aligned}$$

$$\begin{array}{ccc}
\begin{array}{c}
\begin{array}{ccc}
z & \begin{array}{c} \xrightarrow{(g_1, \gamma_1)} \\ \Downarrow (h_1, g_1, \gamma_1) =: a \\ \xrightarrow{t(h_1, g_1, \gamma_1)} \end{array} & y
\end{array}
\end{array}
&
\begin{array}{c}
\begin{array}{ccc}
y & \begin{array}{c} \xleftarrow{(g_2, \gamma_2)} \\ \Downarrow (h_2, g_2, \gamma_2) =: a_2 \\ \xleftarrow{t(h_2, g_2, \gamma_2)} \end{array} & x
\end{array}
\end{array}
&
\mapsto
\begin{array}{c}
\begin{array}{ccc}
z & \begin{array}{c} \xrightarrow{(g_1 g_2, \Phi(1, g_2^{-1}, \gamma_1) \gamma_2)} \\ \Downarrow a_1 \cdot_h a_2 \\ \xrightarrow{(\delta(h_1) g_1 \delta(h_2) g_2, \Phi(\phi((g_1 g_2)^{-1})((h_1 \phi(g_1)(h_2))^{-1}), 1, \gamma))} \end{array} & x
\end{array}
\end{array}
\end{array}$$

It is routine to verify that  $\cdot_h$  is a groupoid morphism. Thus these data define a strict Lie 2-groupoid. It is obvious that this Lie 2-groupoid differentiates to the Lie 2-algebroid (16). This Lie 2-groupoid can thus be interpreted as the “action”-groupoid of the action  $\Phi$ .

Next we remark that, under certain assumptions, the above action  $\Phi$  in Theorem 2.15 defines a *principal 2-group bundle* (over a manifold  $N$ ) in the sense of Wockel [29, Def. I.8]. One reason why principal 2-group bundles are interesting is the following [29, Rem. II.11]: when the Lie 2-group  $\mathcal{G}$  corresponds to a crossed module of Lie groups of the form  $(H, \text{Aut}(H), \partial, \phi)$ , where  $H$  is a Lie group and  $\partial: H \rightarrow \text{Aut}(H)$  is given by conjugation, then principal  $\mathcal{G}$ -2-bundles define gerbes over  $N$  [14].

**Proposition 2.20.** *Let  $\Phi: (H \rtimes G) \times \Gamma \rightarrow \Gamma$  be a Lie 2-group action such that both the  $G$ -action on  $M$  and the  $H \rtimes G$ -action on  $\Gamma$  are free and proper with the same quotient  $N := M/G = \Gamma/(H \rtimes G)$ . Then the action  $\Phi$  makes*

$$\Gamma \xrightarrow{\pi} N$$

*into a principal  $\mathcal{G}$ -2-bundle.*

*Proof.* A special case of the the above statement is [8, Prop. 4.2]. Its proof carries over literally to the present case, after one checks that the projection  $\Gamma \rightarrow \Gamma/(H \rtimes G) = N$  is a Lie groupoid morphism. The latter fact follows easily since the action map  $\Phi: (H \rtimes G) \times \Gamma \rightarrow \Gamma$  is a Lie groupoid morphism.  $\square$

### 3 Examples

In this section we display classes of examples of Lie 2-algebra actions on NQ-1 manifolds (see §1.2) and of the integrated actions (see §2.3).

The starting data for the first example is just a bracket-preserving map of a Lie algebra into the sections of a Lie algebroid.

*Example 3.1.*  $[\mathfrak{g}$  acting on  $A$ ] Let  $\mathfrak{g}$  be a Lie algebra,  $A \rightarrow M$  a Lie algebroid and  $\eta: \mathfrak{g} \rightarrow (\Gamma(A), [\cdot, \cdot]_A)$  a Lie algebra morphism<sup>14</sup>. Then  $(A[1], Q := Q_A)$  is an NQ-1 manifold by Lemma 1.13, and

$$\begin{aligned}
\mu: \mathfrak{g}[1] \oplus \mathfrak{g} &\rightarrow \chi(A[1]) \\
(w, 0) &\mapsto \eta(w) \\
(0, \delta w) &\mapsto [Q, \eta(w)]
\end{aligned}$$

is a morphism a DGLAs. Here  $\mathfrak{g}[1] \oplus \mathfrak{g}$  denotes the strict Lie-2 algebra with  $\delta = Id_{\mathfrak{g}}$  and  $[\cdot, \cdot]$  given by the Lie bracket on  $\mathfrak{g}$  and its adjoint action. Further we view  $\eta(w)$  as an element of  $\chi_{-1}(A[1]) \cong \Gamma(A)$ .

<sup>14</sup>In [20, Def. 6.1] this is called *A-action* of  $\mathfrak{g}$  on  $M$ .

The following example is a special case of Ex. 3.1 for which we can write down explicitly the integrated action  $\Phi$ .

*Example 3.2. [ $\mathfrak{g}$  acting on  $TM$ ]* Let  $G$  be a Lie group and  $\hat{\eta}$  an action of  $G$  on a manifold  $M$  (which for simplicity we take to be simply connected). It is immediate that the product action

$$\begin{aligned} (G \times G) \times (M \times M) &\rightarrow (M \times M) \\ (g_1, g_2), (m_1, m_2) &\mapsto (g_1 m_1, g_2 m_2) \end{aligned} \quad (20)$$

is a Lie 2-group action, where all the Lie groupoids appearing are pair groupoids.

We show how to recover this Lie 2-group action from infinitesimal data. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and  $\eta: \mathfrak{g} \rightarrow \chi(M)$  the infinitesimal action corresponding to  $\hat{\eta}$  (that is, a Lie algebra morphism  $\mathfrak{g} \rightarrow (\Gamma(TM), [\cdot, \cdot]_{TM})$ ). By Ex. 3.1 we obtain a (strict) morphism of DGLAs  $\mu: \mathfrak{g}[1] \oplus \mathfrak{g} \rightarrow \chi(T[1]M)$ . It induces a Lie algebra morphism

$$\begin{aligned} \tilde{\mu}: \mathfrak{g}_{ab} \rtimes \mathfrak{g} &\rightarrow \chi(TM) \\ (w, 0) &\mapsto \eta(w) \\ (0, \delta w) &\mapsto (\eta(w))^T. \end{aligned}$$

Here  $\mathfrak{g}_{ab}$  denotes the vector space underlying the Lie algebra  $\mathfrak{g}$  and the first  $\eta(w)$  is viewed as a vector field tangent to the fibers of  $TM \rightarrow M$ . The tangent lift  $(\eta(w))^T$  of the vector field  $\eta(w)$  on  $M$  appears since it agrees with the element  $[[Q, \eta(w)], \cdot] = [\eta(w), \cdot]_{TM}$  of  $CDO(TM)$  (see Lemma 1.6). By Prop. 2.17 this integrates to the Lie 2-group action

$$\begin{aligned} \Phi: (G \rtimes G) \times (M \times M) &\rightarrow (M \times M) \\ (h, g), (m_1, m_2) &\mapsto (hgm_1, gm_2). \end{aligned}$$

Under the isomorphism of Lie 2-groups (over  $Id_G$ )

$$G \rtimes G \cong G \times G, \quad (h, g) \mapsto (hg, g)$$

to the pair groupoid  $G \times G \rightrightarrows G$ , this action corresponds to (20).

*Example 3.3. [Actions on  $T^*M$ ]* Let  $\mathfrak{h}[1] \oplus \mathfrak{g}$  be a strict Lie-2 algebra and  $(M, \pi)$  a Poisson manifold. Since  $T^*M$  is a Lie algebroid,  $(\mathcal{M}, Q) := (T^*[1]M, [\pi, \cdot]_S)$  is a NQ-1 manifold, where  $[\cdot, \cdot]_S$  denotes the Schouten bracket of multivector fields.

In [8, §8,9] Cattaneo and the first author consider a morphism of DGLAs of the form

$$\begin{aligned} \mathfrak{h}[1] \oplus \mathfrak{g} &\rightarrow \chi_{-1}(\mathcal{M}) \oplus \chi_0(\mathcal{M}) \\ (w, v) &\mapsto X_{J_0^* w} + X_{J_1^* v} \end{aligned} \quad (21)$$

where  $(J_0, J_1): \mathcal{M} \rightarrow (\mathfrak{h}[1] \oplus \mathfrak{g})^*[1]$  is a Poisson (moment) map, and discuss its reduction.

When  $\mathfrak{h} = \mathfrak{g}$  and the differential is  $Id_{\mathfrak{g}}$ , the morphism (21) is recovered from our Ex. 3.1 with  $A = T^*M$  and  $\eta: \mathfrak{g} \rightarrow \Gamma(T^*M), w \mapsto -d(J_0^* w)$ . Notice that in this case, as pointed out in [8, Ex. 15.2], the morphism (21) is equivalent to an ordinary Hamiltonian action of  $\mathfrak{g}$  on  $(M, \pi)$ .

Given a Lie group, we see that the conjugation and adjoint actions fit into our framework:

*Example 3.4.* [Adjoint action] Let  $\mathfrak{g}$  be a Lie algebra and consider the DGLA morphism

$$\mu: \mathfrak{g} \rightarrow \chi(\mathfrak{g}[1]), \quad v \mapsto [Q, \iota_v],$$

where  $\iota_v$  is the (constant) vector field corresponding to  $v$  under  $\mathfrak{g} \cong \chi_{-1}(\mathfrak{g}[1])$  and  $Q$  is the homological vector field on  $\mathfrak{g}[1]$ . We now show that diagram (19) about transformation groupoids/algebroids applied to this example is the following:

$$\begin{array}{ccc} \text{Transf. groupoid of conjugation} & \xrightarrow{\text{Lie}_H} & \text{Transf. groupoid of } Ad \\ & & \downarrow \text{Lie} \\ & & \text{Transf. algebroid of } ad. \end{array}$$

The Lie algebra action on  $\mathfrak{g}$  corresponding to  $\mu$  reads

$$\tilde{\mu} = ad: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) \subset \chi(\mathfrak{g}), \quad v \mapsto ad_v$$

by Lemma 1.6. Integrating this Lie algebra action we obtain essentially the adjoint action of  $G$  on  $\mathfrak{g}$ . More precisely, we obtain the Lie algebroid morphism (see Prop. 2.14)

$$\Psi = Ad: (G \rightarrow G) \times (\mathfrak{g} \rightarrow \{pt\}) \rightarrow (\mathfrak{g} \rightarrow \{pt\}).$$

Integrating this Lie algebroid morphism, by Prop. 2.17 a) we obtain the action of  $G$  on itself by conjugation, or more precisely, the Lie 2-group action

$$\Phi = \text{conjugation}: (G \rightrightarrows G) \times (G \rightrightarrows \{pt\}) \rightarrow (G \rightrightarrows \{pt\}).$$

## A Appendix

In this appendix we prove two statements presented in the main body of the paper.

### A.1 Proof of Lemma 1.6

*Proof of Lemma 1.6 .* By Example 1.4

$$\Gamma(S^\bullet(E^*[-1])) = \Gamma(\wedge^\bullet E^*) \tag{22}$$

is a sheaf over  $M$  making  $\mathcal{M} := E[1]$  into a degree 1 N-manifold. On the right hand side appears the ordinary exterior product of the vector bundle  $E^*$ , and elements of  $\wedge^k E^*$  are assigned degree  $k$ .

Conversely, let  $M$  be a topological space and  $\mathcal{O}_M$  a sheaf of graded commutative algebras over  $M$  as in Def. 1.2, defining a graded manifold  $\mathcal{M}'$ . Then  $M$  must be a smooth manifold and the degree 1 elements of  $\mathcal{O}_M$  form a locally free module over  $C^\infty(M)$ , hence sections of a vector bundle, whose dual we denote by  $E$ . From Def. 1.1 we conclude that  $\mathcal{M}' \cong E[1]$ .

From Def. 1.1 it is clear that  $C(\mathcal{M})$  is generated (as a graded commutative algebra) by its elements of degree 0 and 1. By eq. (22) we have  $C_0(\mathcal{M}) = C^\infty(M)$  and  $C_1(\mathcal{M}) = \Gamma(E^*)$ .

A vector field on  $\mathcal{M}$ , since it is a graded derivation, is determined by its action on functions of degree 0 and 1. Let  $f_0, g_0 \in C_0(\mathcal{M})$  and  $f_1, g_1 \in C_1(\mathcal{M})$ . If  $X_{-1} \in \chi_{-1}(\mathcal{M})$ , it



maps  $C_0(\mathcal{M})$  to zero and maps  $C_1(\mathcal{M})$  to  $C_0(\mathcal{M})$ . Hence the graded derivation property is simply

$$X_{-1}(f_0 g_1) = f_0 X_{-1}(g_1), \quad (23)$$

so the action of  $X_{-1}$  on  $\Gamma(E^*)$  is  $C^\infty(M)$ -linear, i.e. given pairing with a section of  $E$ , and we conclude that  $\chi_{-1}(\mathcal{M}) = \Gamma(E)$ .

If  $X_0 \in \chi_0(\mathcal{M})$ , then the action of  $X_0$  preserves  $C_0(\mathcal{M})$  as well as  $C_1(\mathcal{M})$ . The graded derivation property on generators reads

$$X_0(f_0 g_0) = X_0(f_0)g_0 + f_0 X_0(g_0), \quad X_0(f_0 g_1) = f_0 X_0(g_1) + X_0(f_0)g_1.$$

The first equation tells us that  $X_0|_M$ , defined restricting the action of  $X_0$  to  $C_0(\mathcal{M})$ , is a vector field on  $M$ , and altogether we conclude that  $X_0$  is a covariant differential operator on  $E^*$  with symbol  $X_0|_M$ . Hence  $\chi_0(\mathcal{M}) = CDO(E^*)$ .

The canonical identification  $CDO(E^*) \cong CDO(E)$  obtained dualizing CDOs sends  $X_0 \in \chi_0(\mathcal{M}) = CDO(E^*)$  to  $[X_0, \cdot] \in CDO(E)$  (using the identification  $\chi_{-1}(\mathcal{M}) = \Gamma(E)$ ). Indeed eq. (2) applied to  $Y^* = X_0$  reads

$$X_{-1}(X_0(\xi)) + [X_0, X_{-1}](\xi) = X_0(X_{-1}(\xi))$$

for all  $X_{-1} \in \chi_{-1}(\mathcal{M}) = \Gamma(E)$  and  $\xi \in C_1(\mathcal{M}) = \Gamma(E^*)$ .  $\square$

*Remark A.1.* Consider again a vector bundle  $E \rightarrow M$  and  $\mathcal{M} = E[1]$ . In Lemma 1.6 we saw that  $C_0(\mathcal{M}) = C^\infty(M)$  and that  $C_1(\mathcal{M})$  agrees with the fiber-wise linear functions on  $E$ . [8, Lemma 8.7] states that this induces a canonical, bracket preserving identification of vector fields

$$\begin{aligned} \chi_{-1}(E[1]) &\cong \{\text{vertical vector fields on } E \text{ which are invariant under translations in the fiber direction} \} \\ \chi_0(E[1]) &\cong \{\text{vector fields on } E \text{ whose flow preserves the vector bundle structure} \}. \end{aligned}$$

## A.2 Proof of Prop. 2.14

*Proof of Prop. 2.14.* Checking that  $\Psi$  is really a group action is a straight-forward computation that uses

$$\psi(g, (\mu(w))_x) = (\mu(gw))_{gx} \quad \text{for all } g \in G, w \in \mathfrak{h}, x \in M,$$

which is just the equivariance of  $\mu|_{\mathfrak{h}}: \mathfrak{h} \rightarrow \chi_{-1}(A) = \Gamma(A)$  with respect to the  $G$  actions on  $\mathfrak{h}$  and on the vector bundle  $A$ . The infinitesimal  $\mathfrak{g}$ -equivariance holds by Lemma 1.17, and as  $G$  is connected the global  $G$ -equivariance also holds. A simple computation also shows that differentiating the group action  $\Psi$  one obtains  $\tilde{\mu}$ . This proves the first part of the proposition.

Now we check that  $\Psi$  preserves the anchor maps. Fix  $(w, g) \in \mathfrak{h} \rtimes G$  and  $a_x \in A_x$  (the fiber of  $A$  over  $x \in M$ ). Applying the anchor to  $((w, g), a_x)$  we obtain  $(\overrightarrow{\delta w}(g), \rho_A(a_x))$ , and applying the derivative of the action map  $G \times M \rightarrow M$  gives

$$(\delta w)_M(gx) + g \cdot \rho_A(a_x), \quad (24)$$

where  $(\delta w)_M$  denotes the vector field on  $M$  induced by the infinitesimal action of  $\delta w \in \mathfrak{g}$  on  $M$ , the dot denotes tangent lift of the action of  $G$  on  $M$ , and  $\rho_A$  is the anchor of  $A$ . Now

$$(\delta w)_M = (\mu(\delta w))|_M = [Q, \mu w]|_M = \rho_A(\mu w),$$

where the second equality (between vector fields on  $A[1]$ ) holds because  $\mu$  respects differentials, or alternatively by Lemma 1.17. We saw that the  $G$ -action  $\psi$  on  $A$  is by Lie algebroid automorphism, so in particular  $\rho_A: A \rightarrow TM$  is  $G$ -equivariant and  $g \cdot \rho_A(a_x) = \rho_A(g \cdot a_x)$ . Hence (24) is equal to  $\rho_A(\Psi((w, g), a_x))$ , proving that  $\Psi$  respects the anchor maps.

Checking that  $\Psi$  maps the bracket  $[\cdot, \cdot]_E$  on the product Lie algebroid  $E := (\mathfrak{h} \rtimes G) \times A$  to the bracket  $[\cdot, \cdot]_A$  on  $A$  is more involved. First we remark that  $E$ , as a vector bundle over  $G \times M$ , is a Whitney sum of pullback vector bundles  $\pi_G^*(\mathfrak{h} \rtimes G) \oplus \pi_M^*A$ , where  $\pi_G$  and  $\pi_M$  are the obvious projections of  $G \times M$  onto  $G$  and  $M$ . We define the vector bundle automorphism

$$\varphi: \pi_M^*A \rightarrow \pi_M^*A, \quad a_{(g,x)} \mapsto g^{-1}(a_{(g,x)}) \quad (25)$$

over the base diffeomorphism  $(g, x) \mapsto (g, g^{-1}x)$  of  $G \times M$ . It is actually a Lie algebroid automorphism, since each  $g \in G$  acts by Lie algebroid automorphisms of  $A$ . Notice that any  $a \in \Gamma(A)$  can be pulled back to a section of  $\pi_M^*A \subset E$  (also denoted by  $a$ ), and its image  $\varphi(a)$  under  $\varphi$  is given by

$$(\varphi(a))_{(g,x)} = g^{-1} \cdot a_{gx} \quad (26)$$

The section  $\varphi(a) \in \Gamma(E)$  is  $\Psi$ -projectable, and projects to  $a \in \Gamma(A)$ . Similarly, for any  $w \in \mathfrak{h}$ , the constant section  $w \in \Gamma(\pi_G^*(\mathfrak{h} \times G)) \subset E$  projects to  $\mu(w) \in \Gamma(A)$ . Sections of the form  $\varphi(a)$  and  $w$  span the whole of  $E$ , hence, by [17, Prop. 4.3.8], it suffices to consider such sections. We have

$$\Psi[\varphi(a_1), \varphi(a_2)]_E = \Psi(\varphi[a_1, a_2]_A) = [a_1, a_2]_A = [\Psi\varphi a_1, \Psi\varphi a_2]_A$$

where in the first equality we used that  $\varphi$  is a Lie algebroid automorphism of  $\pi_M^*A$ . We have

$$\Psi[w_1, w_2]_E = \Psi([w_1, w_2]_\delta) = \mu([w_1, w_2]_\delta) = [\mu w_1, \mu w_2]_A = [\Psi w_1, \Psi w_2]_A$$

where in the third equality we used Lemma 1.17.

Next we show that

$$[w, \varphi(a)]_E = \varphi([\mu(w), a]_A) \text{ for all } a \in \Gamma(A), w \in \mathfrak{h}, \quad (27)$$

as it will imply that

$$\Psi[\varphi(a), w]_E = \Psi\varphi([a, \mu(w)]_A) = [a, \mu(w)]_A = [\Psi\varphi(a), \Psi w]_A$$

and thus conclude our proof.

To show (27) we choose, on an open set  $U$  of  $M$ , a local frame of sections  $a_i$  of  $\Gamma(A)$ . We have

$$(\varphi a_i)_{(g,x)} = f_i^j(g, x) a_j(x) \quad (28)$$

for functions  $f_i^j$  defined on  $G \times U$ . Here we use the Einstein summation convention. By the Leibniz rule we can write the l.h.s. of eq. (27) as

$$([w, \varphi(a_i)]_E)_{(g,x)} = \vec{\delta}w(f_i^k(g, x))a_k(x), \quad (29)$$

where  $\vec{\delta}w$  denotes the right-invariant vector field on  $G$  whose value at the identity is  $\delta w$ .

Notice that this lies in  $\pi_M^*A \subset E$ . Further, using the identification  $\Gamma(A) = \chi_{-1}(A[1])$ , we have

$$([\mu w, a_i]_A)_x = [[Q_A, \mu w], a_i]_x = [\mu(\delta w), a_i]_x = (\mathcal{L}_{\mu(\delta w)}a_i)_x = \frac{d}{dt}|_0 \exp(-t\delta w) \cdot (a_i)_{\exp(t\delta w) \cdot x} \quad (30)$$

$$\stackrel{(26)}{=} \frac{d}{dt}|_0 (\varphi a_i)_{(\exp(t\delta w), x)} = \frac{d}{dt}|_0 f_i^j(\exp(t\delta w), x) a_j(x).$$

We deduce that

$$\begin{aligned} (\varphi[\mu(w), a_i]_A)_{(g,x)} &\stackrel{(26)}{=} g^{-1} \cdot ([\mu(w), a_i]_A)_{gx} \\ &\stackrel{(30)}{=} \frac{d}{dt}|_0 f_i^j(\exp(t\delta w), gx) \cdot g^{-1} a_j(gx) \\ &\stackrel{(26)}{=} \frac{d}{dt}|_0 f_i^j(\exp(t\delta w), gx) \cdot (\varphi a_j)_{(g,x)} \\ &= \frac{d}{dt}|_0 f_i^j(\exp(t\delta w), gx) \cdot f_j^k(g, x) a_k(x) \\ &= \frac{d}{dt}|_0 f_i^k(\exp(t\delta w)g, x) a_k(x) \\ &\stackrel{\vec{\delta}w}{=} \vec{\delta}w(f_i^k(g, x)) a_k(x) \\ &\stackrel{(29)}{=} ([w, \varphi(a_i)]_E)_{(g,x)}. \end{aligned}$$

Here in the third last equality we used the “multiplicativity formula”

$$f_i^j(gh, x) = f_i^k(g, hx) f_k^j(h, x) \quad (31)$$

which can be checked writing out  $(\varphi a_i)_{(gh,x)}$  by means of eq. (26). Hence eq. (27) is proved and we are done.  $\square$

*Remark A.2.* It can be checked that the Lie algebroid automorphism  $\varphi$  of  $\pi_M^*A$  in (25) can be extended to a Lie algebroid automorphism of  $E$  by asking that it maps the constant section  $w$  to  $w - \varphi(\mu w) \in \Gamma(E)$  for all  $w \in \mathfrak{h}$ .

## References

- [1] M. Artin and B. Mazur. On the van Kampen theorem. *Topology*, 5:179–189, 1966.
- [2] J. C. Baez and A. S. Crans. Higher-dimensional algebra. VI. Lie 2-algebras. *Theory Appl. Categ.*, 12:492–538 (electronic), 2004.

- [3] J. C. Baez and A. D. Lauda. Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.*, 12:423–491 (electronic), 2004.
- [4] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri. Reduction of Courant algebroids and generalized complex structures. *Adv. Math.*, 211(2):726–765, 2007.
- [5] A. S. Cattaneo. From topological field theory to deformation quantization and reduction. In *International Congress of Mathematicians. Vol. III*, pages 339–365. Eur. Math. Soc., Zürich, 2006.
- [6] A. S. Cattaneo and G. Felder. Poisson sigma models and symplectic groupoids. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 61–93. Birkhäuser, Basel, 2001.
- [7] A. S. Cattaneo and F. Schätz. Introduction to supergeometry. *Rev. Math. Phys.*, 23(6):669–690, 2011.
- [8] A. S. Cattaneo and M. Zambon. A super-geometric approach to Poisson reduction. [Arxiv:1009.0948](https://arxiv.org/abs/1009.0948).
- [9] M. Crainic and R. L. Fernandes. Integrability of Lie brackets. *Ann. of Math. (2)*, 157(2):575–620, 2003.
- [10] E. Getzler. Lie theory for nilpotent  $L_\infty$ -algebras. *Ann. of Math. (2)*, 170(1):271–301, 2009.
- [11] A. Henriques. Integrating  $L_\infty$ -algebras. *Compos. Math.*, 144(4):1017–1045, 2008.
- [12] Y. Kosmann-Schwarzbach. Derived brackets. *Lett. Math. Phys.*, 69:61–87, 2004.
- [13] Y. Kosmann-Schwarzbach and K. C. H. Mackenzie. Differential operators and actions of Lie algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 213–233. Amer. Math. Soc., Providence, RI, 2002.
- [14] C. Laurent-Gengoux, M. Stiénon, and P. Xu. Non-abelian differentiable gerbes. *Adv. Math.*, 220(5):1357–1427, 2009.
- [15] K. C. H. Mackenzie. Double Lie algebroids and the double of a Lie bialgebroid. [arXiv.org:math/DG/9808081](https://arxiv.org/abs/math/9808081), 1998.
- [16] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. II. *Adv. Math.*, 154(1):46–75, 2000.
- [17] K. C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
- [18] R. Mehta and X. Tang. From double lie groupoids to local lie 2-groupoids. *Bulletin of the Brazilian Mathematical Society*, 42(4):651–681, 2011-12-01.

- [19] R. A. Mehta. Supergroupoids, double structures, and equivariant cohomology, PhD thesis, University of California, Berkeley, 2006. [arXiv:math.DG/0605356](#).
- [20] R. A. Mehta.  $Q$ -algebroids and their cohomology. *J. Symplectic Geom.*, 7(3):263–293, 2009.
- [21] R. A. Mehta and M. Zambon.  $L_\infty$ -algebra actions. ArXiv.
- [22] D. Roytenberg. On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 169–185. Amer. Math. Soc., Providence, RI, 2002.
- [23] P. Ševera. Letter to Alan Weinstein. <http://sophia.dtp.fmph.uniba.sk/~severa/letters/no8.ps>.
- [24] P. Ševera. Some title containing the words “homotopy” and “symplectic”, e.g. this one. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 121–137. Univ. Luxemb., Luxembourg, 2005.
- [25] L. Stefanini. On Morphic Actions and Integrability of LA-Groupoids. PhD thesis, [arXiv.math:0902.2228](#), 2009.
- [26] A. Y. Vaĭntrob. Lie algebroids and homological vector fields. *Uspekhi Mat. Nauk*, 52(2(314)):161–162, 1997.
- [27] T. Voronov. Higher derived brackets and homotopy algebras. *J. Pure Appl. Algebra*, 202(1-3):133–153, 2005.
- [28] T. Voronov.  $Q$ -manifolds and Mackenzie theory: an overview. [arXiv.org:0709.4232](#), 2007.
- [29] C. Wockel. Principal 2-bundles and their gauge 2-groups. *Forum Math.*, 23(3):565–610, 2011.
- [30] M. Zambon and C. Zhu. Distributions and quotients on degree 1 NQ-manifolds and Lie algebroids. 02 2012, ArXiv [1202.1378](#).